MAJORITY VOTING IN HIGHER-ORDER SOCIETIES

Draft, 27 January 2020

Abstract

The aim of this paper is to expand the application of social welfare functions (swfs) like for example the simple majority rule μ and the consensus rule κ to the domain of higher-order societies (i.e., societies that have other societies as members). It is shown, first, that on this larger domain swfs may fail to retain some of their standard properties. For example, μ comes to violate anonymity and responsiveness. But, second, in such cases new properties of the swfs are revealed. I focus on the reducibility property of a swf: its capacity to mimic the behavior of other swfs. I give a characterization of the classes of swfs reducible to μ and to κ and prove that while κ is not reducible to μ, conversely μ can be reduced to κ. Finally, I show that some swfs can be extended, in the sense that the application of the swf to a society formed of a large number of members is reducible to iterate applications of that swf to a (higher-order) society formed of at most two members. Some larger implications of the results are discussed.

Keywords: higher-order societies, majority rule, unanimity, consensus, reducibility, binary societies

JEL classification: D71

Social welfare functions (swfs) like the simple majority μ are usually studied on societies consisting in an arbitrary number *n* of members called "individuals". In this paper I shall depart from this assumption by taking into account a larger domain to which a swf applies. It includes not only societies formed of a (finite) number of individuals (first-order societies), but also more complex societies. These societies are higher-order, which means that their members may be individuals, but also other societies. In applying a swf to a profile of such societies, there is no guarantee that it retains the same behavior it had on first-order societies. A straightforward example is that of μ: although it is anonymous on first-order societies, i.e. it is indifferent to the names of individuals in them, it may behave even in a dictatorial way on some societies of societies.

By taking into account such societies we move from direct democracy to the representative (or indirect) democracy. As Murakami (1966) argued, its study requires an appeal to a "hierarchy of voting procedures, each of which may be called a council. Every individual casts a ballot or ballots in one council or councils. A decision in each council is represented in a higher council whose decision is, in turn, represented in a still higher council and so on". A famous example of applying majority voting to higher-order societies is the election of consuls, censors and other magistrates by the Roman *comitia centuriata* (Rousseau: 2002, IV4; Mommsen: 1894, XI; Taylor: 1966; Hall: 1964). The reform of Servius (the sixth Roman king) divided the population of Rome into six classes, which together comprised 193 *centuriae* (centuries). The assembly was extremely favorable to the aristocracy: the first class alone (consisting in the richest members of the society) had 98 centuries, more than the majority of 97 required for electing a magistrate. The decisions of *comitia centuriata* were not based simply on the votes of the Roman citizens, but also appealed to the group vote of the centuries. The vote of a century was determined by the votes of its members, by use of the simple majority rule. In a second step the assembly decisions were reached by an absolute majority of centuries, each century disposing of one vote in the assembly[1].

[1] Voting was sequential, starting with the vote of a century (perhaps selected by lot) belonging to the first class, and the results were communicated to all the members of the assembly; thus, majorities could be known before the vote of the centuries belonging to other classes (therefore, the recording of their vote was not even completed). As Rousseau observed, the effect of this voting procedure was that what the minority had decided passed for a decision of the multitude. In other words, individual votes were not equal.

The study of swfs is usually guided by the characterization approach. The idea is to find out the class of swfs that satisfy a certain collection of (usually desired or intuitive) properties. A swf is characterizing by giving a set of properties it and only it satisfies. May (1952) is the classic. It proves that the simple majority rule μ is the only swf which satisfies the three properties anonymity, neutrality and responsiveness. Following this approach, a large number of new characterizations of μ were provided. Some of them even appealed to properties involving second-order societies: weak path independence in Asan and Sanver (2002), reducibility in Quesada (2013a), reducibility to subsocieties in Woeginger (2003), which inspired Alcantud (2019).

Murakami (1966; 1968) attempted to account for a swf in an indirect, comparative way. He argued that starting with some swf *g*, we can produce the collection of swfs that in some sense can be reduced to that swf. Since the simple majority rule μ is well-known, if some other swf (the weighted majority rule, for example) is reducible to it, then we can better grasp how it works. Fishburn (1971) and Fine (1972) succeeded in identifying the class of swfs that can be expressed in terms of iterated applications of the simple majority rule μ. Following their account, in this paper I define a set of necessary and sufficient conditions for reducing a swf to the consensus rule κ. I also show that this set includes μ: the simple majority rule can be reduced to κ. In these proofs the appeal to higher-order societies is essential.

These results are technical, but I believe that they have some more general implications. The received view on the relations between unanimity / consensus and simple majority is that they are distinct rules and have different contexts of application. While unanimity and consensus are properly used in constitutional matters, the majority rule (both simple and absolute) is to be applied at the post-constitutional level. Theorem 3 below entails that this distinction is not sharp, because majority decision-making can be explained as a sophisticated consensual decision.

A final point of departure from the standard approach of studying swfs I make in this paper is this. Usually, no special interest is given to the number of individuals in the societies swfs apply to. One only mentions that these societies consist in an arbitrarily chosen number of individuals. However, it is possible to show that reducibility may work when we start with the binary component of a swf, i.e. with its applications to societies formed of only two members. I prove that many swfs can be extended, in the sense that applying them to a society formed of an arbitrarily large number of members can be reduced to iterated applications of their restriction to societies formed of at most two members. Swfs like μ can be extended, but other swfs (the consensus rule κ and the absolute majority rule α among them) cannot.

The paper is structured as follows. In section 1 I present the framework. Some examples of properties of swfs that involve higher-order societies are discussed in section 2. It also includes the main lemma, which is essential in the proofs of theorems. In sections 3 and 4 I present the main results of this paper. I show, first, that although the consensus rule κ is irreducible to μ, the converse relation holds: μ is definable in terms of iterated applications of κ. Then I prove that the class of all κ-reducible swfs is characterized by a set of intuitive properties: neutrality, monotonicity and strong Pareto. In section 4 I use the same approach to show that some swfs can be extended. A by-product of this result is that to characterize a swf we can focus on its binary part. A characterization of the binary majority rule μ^2 which appeals to extremely weak properties is given. All proofs are collected in section 5. Section 6 concludes.

1. The framework

Let $G = \{v_1, v_2, \ldots v_n \ldots\}$ be an infinite group of individuals. For the purposes of this paper the exact nature of these "individuals" is not important. They can be persons, firms or other sorts of entities. The class Ω_G of societies is given by the following recursive procedure[2]:

a) If $S \subseteq G$ and S is finite, then $S \in \Omega_G$;
b) if $S_1 \in \Omega_G$ and $S_2 \in \Omega_G$, then $\{S_1, S_2\} \in \Omega_G$;
c) if $S_1 \in \Omega_G$ and $S_2 \in \Omega_G$, then $S_1 \cup S_2 \in \Omega_G$;
d) if $S \in \Omega_G$, then $\cup S \in \Omega_G$.

A first-order society is a (possibly empty) subset of S. The set of first-order societies will be denoted by Ω^1_G. Note that by construction all societies are finite. Say that a society S' is nested in S if $S' \in S$ or there is a finite sequence $S_1, S_2 \ldots S_m$ such that $S' \in S_1 \in S_2 \in \ldots \in S_m \in S$. The basis $B(S)$ of the society S is the set of all individuals who occur in S or in some society nested in it. If $S = \{v_1, ,v_2 \ldots v_n\}$, I shall denote by S^{-j} the result of removing v_j from S ($j = 1, 2, \ldots n$); with some abuse of notation, if $S = \{S_1, S_2, \ldots S_n\}$, I shall also denote by S^{-j} the result of removing S_j from S ($j = 1, 2, \ldots n$). Say that a society S is binary if it has at most two members and all societies nested in it have at most two members.

The set of alternatives is $\{x, y\}$, with $x \neq y$. A profile for G is a function p: $G \rightarrow \{1, 0, -1\}$ assigning a preference $p(v_j) = p_{v_j}$ to each individual v_j in G. If the number p_{v_j} is 1, x is preferred by v_j to y; if -1, y is preferred by v_j to x; and if 0, then v_j is indifferent between x and y. Say that an individual v_j is concerned if $p_{v_j} \neq 0$. A profile of G is a structure $p = (p_{v_1}, p_{v_2}, \ldots p_{v_n} \ldots)$. For each profile p and society S, the restriction of p to $B(S)$ is denoted by p_S. $\mathbf{P}|_S$ denotes the set of all profiles

[2] To use the set-theoretical parlance, all societies are well-founded. Although non-well-founded societies are possible, I shall not discuss them in this paper.

of S. If p_S^k and $p_S^{k'}$ are profiles of S, then write $p_S^k \leq p_S^{k'}$ if and only if $p_{v_j}^k \leq p_{v_j}^{k'}$ for all $v_j \in B(S)$. Write also $p_S^k < p_S^{k'}$ if $p_S^k \leq p_S^{k'}$ and $p_{v_j}^k < p_{v_j}^{k'}$ for at least a $v_j \in B(S)$. For each profile p_S^k I shall denote by - p_S^k for the profile $p_S^{k'}$ with the property that in it all the preferences of the individuals in p_S^k are reversed: $p_{v_j}^k = - p_{v_j}^{k'}$ for all v_j in S.

A social welfare function (swf) is a mapping f: $\mathbf{P}|_S \rightarrow \{1, 0, -1\}$ for each society $S \in \Omega_G$. For each profile p_S of a society S, f gives the collective preference of its members over the alternatives x and y. Applying a swf f to profiles of societies is an iterative process: if $S = \{v_1, v_2 \dots v_m\}$ is a first-order society, then $f(p_S) = f(p_{v_1}, \dots p_{v_m})$; and if $S = \{S_1, \dots S_m\}$ is a higher-order society, i.e. its members are also societies, then μ is defined by: $f(p_S) = f(f(p_{S_1}), \dots f(p_{S_m}))$.

I introduce eight voting rules. They are given for first-order societies, but their definition can be easily extended to all societies in Ω_G.

- The **unanimity** rule υ: $\upsilon(p_S) = 1$ if $p_{v_j} = 1$ for all $v_j \in S$; $\upsilon(p_S) = -1$ if $p_{v_j} = -1$ for all $v_j \in S$; and $\upsilon(p_S) = 0$ in all the other cases.
- The **consensus** rule κ: $\kappa(p_S) = 1$ if $p_{v_j} \geq 0$ for all $v_j \in S$ and $p_{v_j} = 1$ for some $v_j \in S$; $\kappa(p_S) = -1$ if $p_{v_j} \leq 0$ for all $v_j \in S$ and $p_{v_j} = -1$ for some $v_j \in S$; and $\kappa(p_S) = 0$ in all the other cases.
- The (**simple**) **majority** rule μ: $\mu(p_S) = \text{sgn}(\sum_{v_j \in S} p_{v_j})$. The sgn function is given by: if $k > 0$, then $\text{sgn}(k) = 1$; if $k < 0$, then $\text{sgn}(k) = -1$; and if $k = 0$, then $\text{sgn}(k) = 0$.
- The **weighted majority** rule μ_w: let w be a function that attaches an integer $a_j > 0$ to each voter v_j. Then: $\mu_w(p_S) = \text{sgn}(\sum_{v_j \in S} a_j p_{v_j})$.
- The **absolute majority** rule α: $\alpha(p_S) = 1$ if $|\{v_k \in S\colon p_{v_k} = 1\}| > n/2$; $\alpha(p_S) = -1$ if $|\{v_k \in S\colon p_{v_k} = -1\}| > n/2$; and $\alpha(p_S) = 0$ in all the other cases.
- The **chairperson tie-breaking** rule ch: if $v_1 \in S$ is the chairperson, then $ch(p_S) = 1$ if $\mu(p_S) = 1$; $ch(p_S) = -1$ if $\mu(p_S) = -1$; and $ch(p_S) = p_{v_1}$ if $\mu(p_S) = 0$.
- The **minimum** rule Min: $\text{Min}(p_S) = \min(p_{v_1}, \dots p_{v_n})$
- The **maximum** rule Max: $\text{Max}(p_S) = \max(p_{v_1}, \dots p_{v_n})$

In a non-technical language, given an agenda consisting in exactly two alternatives *x* and *y*, the unanimity rule υ selects *x* if all the members of the group strictly prefer *x* to *y*. The consensus rule κ selects *x* if no member of the group is against it (i.e. no voter strictly prefers *y*) and at least a member of the group strictly prefers *x*. The simple majority rule μ selects *x* if the members of the group who strictly prefer it are more numerous than the members of the group who strictly prefer *y* (and similarly for *y*). In all the cases when the corresponding conditions are not satisfied, the three rules give group indifference. The weighted majority rule differs from the simple majority in that voters are assigned different numbers of votes. By the absolute majority rule α an alternative is preferred if more than half of the members of the society strictly prefer it. According to *ch*, the group preference is decisive except in cases when, by the simple majority rule, the group is unconcerned; in that case the chairperson makes the final choice. The maximum rule Max takes the most favorable individual preference for *x*, and the minimum rule Min takes the least favorable individual preference for it.

I shall use superscripts to indicate the number of members of the society *S* to which the swf *f*. Thus, f^k indicates that the rule applies to profiles of a society formed of *k* members; in particular, f^2 applies to profiles of a society formed of just two members. Note that if *S* has exactly two members, then for all profiles p_S the absolute majority rule α coincides with the unanimity rule υ, and the simple majority rule μ coincides with the consensus rule κ:

$$\upsilon^2(p_S) = \alpha^2(p_S)$$

$$\mu^2(p_S) = \kappa^2(p_S)$$

The following matrixes describe this situation

Matrix 1a

$\mu^2(p_{\{v_1,v_2\}}) =$ $\kappa^2(p_{\{v_1,v_2\}})$		p_{v_2}		
		1	0	-1
p_{v_1}	1	1	1	0
	0	1	0	-1
	-1	0	-1	-1

Matrix 1b

$\upsilon^2(p_{\{v_1,v_2\}}) =$ $\alpha^2(p_{\{v_1,v_2\}})$		p_{v_2}		
		1	0	-1
p_{v_1}	1	1	0	0
	0	0	0	0
	-1	0	0	-1

Swfs may satisfy certain properties. Here is a collection of properties I shall appeal to below:

- **Strong Pareto** (**SP**). If p_S is such that at it $p_{v_j} \geq 0$ for all $v_j \in S$ and $p_{v_j} = 1$ for some $v_j \in S$, then $f(p_S) = 1$.
- **Monotonicity** (**Mon**). For any society S and any two profiles p_S^1 and p_S^2 of it, if $p_S^1 \geq p_S^2$, then $f(p_S^1) \geq f(p_S^2)$.
- **Neutrality** (**Neu**). For any society S and any profile p_S of it, $f(p_S) = -f(-p_S)$.
- **Anonymity** (**A**). For any society S and any two profiles p_S^1 and p_S^2 of it, where the voters' preferences in p_S^2 are a permutation of the voters' preferences in p_S^1, $f(p_S^1) = f(p_S^2)$.
- **Responsiveness** (**R**). For any society S and any two profiles p_S^1 and p_S^2 of it, if $p_S^2 > p_S^1$ and $f(p_S^1) \geq 0$, then $f(p_S^2) = 1$.
- **Non-zigzaggedness** (**NZ**). There is no zigzag sequence[3] p_S^1, p_S^2, … p_S^m of profiles of S such that: 1) $f(p_S^k) = 0$ for each k ($k = 1, \ldots n$); and 2) $p_S^1 = - p_S^m$.

---

[3] A sequence p_S^1, p_S^2, … p_S^m of profiles of a society S is zigzag if $p_S^k > p_S^{k+1}$ or $p_S^{k+1} > p_S^k$ for each k ($k = 1, \ldots m - 1$). The property **NZ** was introduced in Fine (1972).

Given a set **X** of properties of swfs, I shall denote by $\Phi_{\mathbf{X}}$ the collection of all swfs that satisfy **X** on Ω^1_G.

Finally, I introduce a core notion. Say that a swf *f* is reducible to a swf *g* if for each society *S* there is some society σ_S such that 1) $B(\sigma_S) \subseteq B(S)$; and 2) for all profiles p_S of *S* we have: $f(p_S) = g(p_{\sigma_S})$. In this case I shall also say that *f* can be expressed in terms of *g*. Sections 3 and 4 of the paper will focus on applications of this notion.

To make the presentation simpler, all the proofs of the theorems and lemmas will be collected in Appendix 1.

2. Moving to higher-order domains

A striking fact is that it is possible for a swf *f* to satisfy a property when applied to profiles of first-order societies but fail to satisfy it in the case of higher-order societies[4]. Take for example Responsiveness (**R**). The majority rule μ satisfies it on the set Ω^1_G of first-order societies, and was used by May (1952) to characterize it. But take the second-order society $S = \{\{v_1, v_2\}, v_3\}$. Suppose that at p^1_S we have: $p_{v_1} = p_{v_2} = -1$ and $p_{v_3} = 1$. Then $\mu(p^1_S) = 0$. Next, suppose that at p^2_S we have: $p_{v_1} = 0$, $p_{v_2} = -1$ and $p_{v_3} = 1$. This profile differs from p^1_S in that v_1 became more favorable to the alternative *x*. We get again $\mu(p^2_S) = 0$. But $p^2_S > p^1_S$ and **R** entails that $\mu(p^2_S) = 1$ – contradiction. So, in this case μ violates **R**.

In his characterization of μ May also appealed to Anonymity (**A**). But μ does not satisfy **A** on higher-order domains. To see this, take the example of the society $S = \{i_1, \{i_1, i_2\}, \{i_1, i_3\}\}$. Observe that for each profile p_S of *S* the preferences of the voters i_2 and i_3 count only if the voter i_1 is unconcerned; in all the other cases the society's preference is exactly the preference of i_1. So i_1 enjoys a special status in *S* and consequently Anonymity fails.

Another difference in the behavior of a swf when applied to first-order and higher-order societies is this. Let me introduce the concept of *f*-synonymy: two societies S_1 and S_2 are *f*-

[4] This approach relies on a number of assumptions: first, the same individual voter is allowed to be a member of different societies in the hierarchy. Secondly, an individual voter votes the same way in every society she is a member of. Third, the councils may be of different sizes or complexity. However, these assumptions may be questioned. It is known that in mass elections the constituencies are disjoint, i.e. no voter is allowed to belong to more than one society. Moreover, an individual voter may vote differently in different committees (Sengupta: 1974). Third, the appeal I shall make to very complex higher-order societies may look unintuitive and awkward. Although second-order societies are easy to manage, it is difficult to make sense of more complex ones. I agree with this observation; my use of extremely complex higher-order societies is motivated only by their usefulness in proving technical results.

synonymous, and write $S_1 \approx_f S_2$ for this, if for each profile p_G it holds that $f(p_{S_1}) = f(p_{S_2})$. The f-synonymy relation is very strong. We can prove that a large class of swfs have the property that in the case of first-order societies f-synonymy is an individuating criterion: any two f-synonymous societies are identical. In other words, if two societies are different, then there is a profile in which f gives different values for the two societies.

Theorem 1. Suppose that the swf $f \in \Phi_{\{\mathbf{UC},\, \mathbf{IISo}\}}$ (i.e. f satisfies the two properties **UC** and **IISo**). If S_1 and S_2 are first-order societies, then $S_1 \approx_f S_2$ if and only if $S_1 = S_2$.

Unconcernedness (**UC**). If $p_{v_i} = 0$ for all $v_i \in S$, then $f(p_S) = 0$.

Independence of Indifferent Societies (**IISo**). If $f(p_S) = 0$ and $v_j \notin S$, then $f(p_{S\cup\{v_j\}}) = p_{v_j}$.

We can immediately check that μ is in $\Phi_{\{\mathbf{UC},\, \mathbf{IISo}\}}$, and so if S_1 and S_2 are in Ω_G^1, then $S_1 \approx_\mu S_2$ implies $S_1 = S_2$. However, this implication does not hold for higher-order societies. For example, we have that $\{\{v_1, v_2\}, v_1\} \approx_\mu \{v_1, \{v_1\}, v_2\}$ (Batra and Pattanaik 1972, fn. 8), which means that two different higher-order societies can still be μ-synonymous. Thus, on higher-order domains the simple majority rule fails to remain an identifying criterion.

In this paper the following lemma has a pivotal role. The four properties it proves focus on sets of subsocieties resulting from S by removing one of its members[5]. Part (a) states that if at S the rule μ favors some alternative, then at least one subsociety of S favors that alternative and no subsociety favors the opposing one. By (b), if S is unconcerned, then the number of subsocieties favoring one alternative equals the number of subsocieties favoring the opposing one. Part (c) states that instead of asking the n voters in S, one could equivalently ask the n possible societies S^{-j}, each formed of n - 1 voters, and form the aggregate preference of their aggregate preferences[6]. The property described in (d) states that we can take only $n - 1$ of these subsocieties, form their aggregate preference and finally aggregate it with the preference of the remaining subsociety, as an equivalent of simply aggregating the preferences of the n voters in S.

5 Note that although S is taken as a first-order society in the formulation of the lemma, in fact the result can be generalized to the cases when its members are societies.

6 Woeginger (2003) called this property "Reducibility to Subsocieties", and took it as an axiom in his characterization of μ.

Lemma 1. Let $S = \{v_1, v_2, \dots v_n\}$, with $n \geq 2$. Put: $\Gamma = \{S^{-1}, S^{-2}, \dots S^{-n}\}$ and $\Gamma^{-j} = \Gamma - \{S^{-j}\}$. Then:

a) If $\mu(p_S) = 1$, then $\mu(p_{S^{-j}}) \geq 0$ for each $j = 1, \dots n$ and $\mu(p_{S^{-j}}) = 1$ for some j. If $\mu(p_S) = -1$, then $\mu(p_{S^{-j}}) \leq 0$ for each $j = 1, \dots n$ and $\mu(p_{S^{-j}}) = -1$ for some j.

b) If $\mu(p_S) = 0$, then $|\{S^{-j}: \mu(p_{S^{-j}}) = 1\}| = |\{S^{-j}: \mu(p_{S^{-j}}) = -1\}|$.

c) $\mu(p_S) = \mu(p_\Gamma)$.

d) $\mu(p_S) = \mu(p_{\{\Gamma^{-j}, S^{-j}\}})$.

3. Reducing swfs

This section and the next one form the core of the paper. They are dedicated to applications of the notion of reducibility of a swf to another swf. Remind that by definition a swf f is reducible to a swf g (or: f can be expressed in terms of g) if for each society S there is some society σ_S such that $B(\sigma_S) \subseteq B(S)$ and for all profiles p_S of S we have: $f(p_S) = g(p_{\sigma_S})$. This means that one always gets the same results if she applies f to a (first-order) society or iteratively applies g to a more complex society. Two types of results will be discussed. First, we may try to establish the collection of all swfs reducible to a given swf. In this section this will be proved for the simple majority rule μ and the consensus rule κ. Second, we can compare the usual formulation of a swf, which applies to a society formed of an arbitrary number of n members, with its binary restriction, i.e. its applications to societies formed of only two members. In the next section I shall prove that for some swfs the n-ary case can be reduced to the binary one. The simple majority rule μ, among others, has this property.

Nearly half of a century ago Murakami (1966; 1968), Fishburn (1971) and Fine (1972) proved that the simple majority rule μ has the amazing capacity to reduce a large collection of swfs. Their main results are expressed in the following theorem[7].

Theorem 2.

a) A swf f is reducible to μ if and only if $f \in \Phi_{\{\mathbf{Neu}, \mathbf{Mon}, \mathbf{NZ}\}}$.

b) If G includes two ideal voters $\mathbf{1}^+$ and $\mathbf{1}^-$, then f is reducible to μ if and only if $f \in \Phi_{\{\mathbf{Mon}\}}$.

[7] The two parts of the theorem are presented in Fine (1972). Fishburn (1971) appeals to a rather different property (his condition 7) to prove part (a).

The "ideal" voters $\mathbf{1}^+$ and $\mathbf{1}^-$ are characterized by the fact that their preferences are profile-independent: for each profile p_S we have $p_{\mathbf{1}^+} = 1$ and $p_{\mathbf{1}^-} = -1$. The voter $\mathbf{1}^+$ always prefers x to y and the voter $\mathbf{1}^-$ always prefers y to x. We can for example imagine[8] that whenever the proposals concern a choice between privatize/keep in state property, there are people who, given their strong ideological commitments, always vote 1 and people who always vote -1. If G includes the two ideal voters, then the absolute majority rule α, the unanimity rule υ and the consensus rule κ are reducible to μ, because they all are monotonic. In Appendix 1 I show as an example how α can be defined in terms of μ on this extended domain[9]. However, if ideal voters are not allowed, α and κ cannot be reduced to μ[10].

I shall give three simple examples to make an idea of how this reducibility works. Observe that given a first-order society S we start with, the new society σ_S we construct is a higher-order one.

Example 1. The weighted majority rule μ_w can be expressed in terms of μ. Consider a small society $S = \{v_1, v_2, v_3\}$ where v_3 has one vote, v_2 has two votes and v_3 has three votes. We construct a new society $\sigma_S = \{v_1, v_2, \{v_2\}, v_3, \{v_3\}, \{\{v_3\}\}\}$ with six members. Clearly, $\mu_w(p_S) = \mu(p_{\sigma_S})$ for all profiles of S.

Example 2. Consider the chairperson tie-breaking rule ch. Let $v_1 \in S$ and put $\sigma_S = \{v_1, S, \{S\}\}$. We can immediately check that ch is reducible to μ, i.e. for all profiles p_S of S it holds that $ch(p_S) = \mu(p_{\sigma_S})$. For if $\mu(p_S) = 1$, then $\mu(p_{\sigma_S}) = \mu(p_{v_1}, \mu(p_S), \mu(\mu(p_{\{S\}}))) = \mu(p_{v_1}, 1, \mu(1)) = \mu(p_{v_1}, 1, 1) = 1$; and if $\mu(p_S) = 0$, then $\mu(p_{\sigma_S}) = \mu(p_{v_1}, 0, \mu(0)) = \mu(p_{v_1}, 0, 0) = \mu(p_{v_1}) = p_{v_1}$. Quesada (2013b) called ch "majority rule with a chairman" and provided a set of axioms to characterize it.

---

[8] An amusing example of such an "ideal" voter was noticed a few years ago by *Der Spiegel*. Dan Dumitru Zamfirescu, a Romanian member of the European Parliament (in the period 2013 – 2014), always voted "Yes", even when the proposals were mutually contradictory. He voted 541 times "Yes". See http://www.spiegel.de/international/europe/suspicious-voting-record-of-romanian-mep-dumitru-zamfirescu-a-929117.html.

[9] This is a special way of relating the simple and the absolute majority rules. In general, as remarked by Sanver (2009), relativism and absolutism are essentially incompatible conceptions of majoritarianism.

[10] Fine (1972) argued that the reason for this is that κ also does not satisfy the non-zigzaggedness property. Fine gives the following example: if the group has exactly three members, then the sequence: 11-1; 1-1-1; 1-11; -1-11 is zigzag, while the value of κ for all the members of the sequence remains unchanged. The same conclusion is in Fishburn (1971). Consider Fishburn's condition 7. Let D^1 = 11-1; D^2 = 1-11; D^3 = -111. We have $D^1 > -D^2$, $D^2 > -D^3$, $D^3 > -D^1$. We get $\sum_{k=1}^{3} D_1^k = 2 > 0$, while $\kappa(D^i) = 0$ for all i – in contradiction with his Condition 4.

The proof that *ch* is reducible to majority gives a simpler and even more intuitive explanation of the connection between it and μ.

Example 3. Suppose that the Parliament P of a state is bicameral: it consists in the Chamber of Deputies *D* and the Senate *S*. P has the following rule, call it ρ: a law is passed if it is voted by both chambers; but if the two chambers have opposing resolutions, then to pass the law a vote of the joined chambers is required[11]. Starting with P, construct a more complex society σ_P as follows: $\sigma_P = \{\{D, S\}, \{\{D, S\}\}, D \cup S\}$. Clearly, we have: $\rho(p_P) = \mu(p_{\sigma_P})$.

Now suppose that the rule ρ is modified as follows. We want to give an additional voting power to the members of the Senate S: when the two chambers are joined, each senator is attached a number $a \geq 1$ of votes (in the simplest case, two votes). Appealing to the example 1 above, a society σ_P can then be easily constructed. Call *Ar* this new rule. Its name is a tribute to Aristotle: as formulated here, it is a simple version of a general rule formulated by Aristotle in his *Politics* (VI 3, 1317a–1318b) in his attempt to describe collective decisions in a democratic polis (Miroiu and Partenie: 2019). Here S corresponds to the class of rich members of a polis, and D to the class of poor members of it.

The main results of this section are expressed by theorems 3 and 4 below:

Theorem 3. The simple majority rule μ is reducible to the consensus rule κ.

Theorem 3 entails that a result proved by Fine (1972) can be modified to obtain a much more important one: a characterization of the swfs reducible to the consensus rule *κ* alone. Fine showed (see his theorem 4) that a swf *f* is reducible to a combination of the simple majority rule and the consensus rule if and only if it is neutral, monotonic and satisfies the positive strong Pareto property. However, since Theorem 3 holds we get:

Theorem 4. A swf *f* is reducible to the consensus rule κ if and only if $f \in \Phi_{\{\mathbf{Neu}, \mathbf{Mon}, \mathbf{SP}\}}$[12].

Remark. Theorem 3 helps us compare theorems 2 and 4. They define two classes of swfs: those in $\Phi_{\{\mathbf{Neu}, \mathbf{Mon}, \mathbf{NZ}\}}$ are reducible to μ and those in $\Phi_{\{\mathbf{Neu}, \mathbf{Mon}, \mathbf{SP}\}}$ are reducible to κ. They differ because as we already noted κ does not satisfy **NZ** and so it is not reducible to μ. But we can easily

[11] This rule was used in the Romanian Parliament in the period 1991 – 2003.

[12] Given neutrality, we can replace the strong Pareto property with its positive part.

see that if **Neu** and **Mon** hold, then **NZ** entails **SP** (see Appendix 1 for this). So, the class of of swfs reducible to μ is included in the class of swfs reducible to κ:

$\Phi_{\{\mathbf{Neu}, \mathbf{Mon}, \mathbf{NZ}\}} \subseteq \Phi_{\{\mathbf{Neu}, \mathbf{Mon}, \mathbf{SP}\}}$.

By theorem 4 all swfs satisfying the three standard conditions Neutrality, Monotonicity and Strong Pareto are proved to be expressible in terms of the consensus rule κ: instead of appealing to them we may construct some new higher-order society and apply κ to its profiles. Theorem 3 highlights a special case: it shows that although κ is not reducible to μ, conversely μ is reducible to κ; therefore, by focusing on a more complex society and appealing to κ we get the same collective preference as applying μ to a simpler first-order society.

However, the received view in the history of political thought is that unanimity/consensus and majority are fundamentally distinct aggregation rules. They have quite different logical structures and have different proper applications. The standard argument runs as follows. First, concerning consensus: "The early theorists (Hobbes, Althusius, Locke, and Rousseau) did assume consensus in the formation of the original contract. They did so because the essence of any contractual arrangement is *voluntary* participation, and no rational being will voluntarily agree to something which yields him, in net terms, expected damage or harm" (Buchanan, Tullock 1999, 248 – 249). The unanimity and consensus rules have their defining application at the constitutional stage. But, second, majority is located in another set of circumstances. "Excepting this original contract, the vote of the majority always binds all the rest" (Rousseau 2002, Bk. 4, Ch. 2, 229 – 230). And Locke: "When any number of men have so consented to make one community or government, they are thereby presently incorporated, and make one body politic, wherein the majority have a right to act and conclude the rest" (Locke 2003, § 95, p. 142).

Rousseau famously discussed this issue with respect to the decision procedures used by the main political institutions (notably the Diet) in his days Poland. He argued that the *liberum veto*[13] "would be less unreasonable if it fell uniquely on the fundamental points of the constitution". It is legitimated by the natural right of societies. It requires unanimity "for the formation of the body politic and for the fundamental laws that pertain to its existence". But when one has not to do with such genuinely fundamental laws, other decision procedures are much more appropriate. Different types of (super)majorities can be used to pass legislation or to decide on important matters of state; simple plurality (our rule μ, when only two alternatives are available) is sufficient in cases of elections and other routine and momentary business, which depend on the "vicissitude of things" (Rousseau 2005, 203 – 204). In all these cases, it is argued, it is not practical to appeal to unanimity or consensus.

[13] This principle applies to both unanimity (our rule υ) and consensus (our rule κ).

However, Theorem 3 shows that consensus and simple majority are not different in nature: simple majority is reducible to consensus. To vote by majority is just to vote by consensus in a more complex way. Applying of μ to a first-order society is equivalent to numerous iterative applications of κ to diverse nested higher-order societies. Instead of an iterate vote by consensus in an extremely intricate and artificially constructed society, the use of a just one step voting by the majority rule μ in a first-order society is much simpler, more intuitive and less costly.

4. Extending swfs

In this section I focus on another application of the notion of reducibility of a swf to another swf. Say that f can be extended (to the n-ary case) when f^n can be expressed in terms of f^2. Some swfs can be extended, while others fail to satisfy this property. A first example concerns associative swfs. A swf f is associative if $f(p_{\{i_1, i_2, \ldots i_{n-1}, i_n\}}) = f^2(f^{n-1}(p_{\{i_1, i_2, \ldots i_{n-1}\}}), p_{i_n})$ for each $S = \{i_1, i_2, \ldots i_n\}$. The following proposition is immediate:

Theorem 5. If f is associative, then f^n can be expressed in terms of f^2.

The swfs Min^2, Max^2 and υ^2 are associative, therefore they can all be extended to the n-ary case. However, some swfs cannot be extended: we cannot find any procedure to define them in terms of their binary parts. Two examples are the absolute majority rule α and the consensus rule κ:

Theorem 6. The binary swfs α^2 and κ^2 cannot be extended.

Clearly, the simple majority rule μ is not associative. Therefore, *prima facie* it cannot be extended. Nevertheless, it is possible to devise a more complex procedure to show that μ^n can be expressed in terms of μ^2. This is the content of Theorem 7 below.

Theorem 7. μ^2 can be extended to μ^n.

Corollary: the fact that $\mu^2 = \kappa^2$ and theorem 7 together entail theorem 3.

One implication of Theorem 7 is that since μ^n can be defined in terms of μ^2, to characterize the simple majority rule μ we only need to do this for its binary part. Below I give such a very weak characterization of μ^2.

Theorem 8. The binary simple majority rule μ^2 is the only swf which satisfies the following four axioms:

Faithfulness (**F**). $f^1(p_{\{v_k\}}) = p_{v_k}$.

Binary Unanimity (**BU**). If $f^1(p_{\{v_1\}}) = f^1(p_{\{v_2\}}) = a \in \{1, -1\}$, then $f^1(p_{\{v_1,v_2\}}) = a$.

Simple Equal Treatment (**SET**). If $f^1(p_{\{v_1\}}) = a \in \{1, -1\}$ and $f^1(p_{\{v_1\}}) = -f^1(p_{\{v_2\}})$, then $f^2(p_{\{v_1,v_2\}}) = 0$.

Independence of Indifferent Singletons (**IIS**). If $f^1(p_{\{v_1\}}) = 0$, then $f^2(p_{\{v_1,v_2\}}) = f^1(p_{\{v_2\}})$.

The axioms are inspired from Xu and Zhong (2010), but are much weaker than theirs. Axiom **F** connects the value of the society $\{v_k\}$ formed of just one individual i_k with the preference of this individual: by **F**, the society must follow the preference of its member. The other axioms display a quite different logical form: rather than connecting individual preferences with social preferences, they connect the values of the swf at distinct societies. By **BU** if two societies are singletons and they prefer the same alternative, then their union will have the same preference. **SET** states that if they have opposite preferences, then their union must be unconcerned. Finally, **IIS** states that if one of them is unconcerned, then the preference of the union depends only on the preference of the other society (we may also note that **IIS** is weaker than the axiom **ISSo** defined in section 2 above).

Observe that when **F** holds the other three axioms take a much simpler (and more intuitive) form:

BU*. If $p_{v_1} = p_{v_2} = a \in \{1, -1\}$, then $f^1(p_{\{v_1,v_2\}}) = a$.

SET*. If $p_{v_1} = -p_{v_2} \neq 0$, then $f^2(p_{\{v_1,v_2\}}) = 0$.

IIS*. If $p_{v_1} = 0$, then $f^2(p_{\{v_1,v_2\}}) = p_{v_2}$.

For example, the axiom **SET*** states that if two concerned individuals v_1 and v_2 have opposite preferences, then the society consisting in the two individuals is unconcerned.

A final note: in line with Xu and Zhong (2010), this characterization of μ^2 appeals to properties that do not allow for changes in the individual preferences: they appeal to a single profile, but admits moves from one society to another. It differs from other characterization like May (1952) and Campbell and Kelly (2000), who considered a single society but multiple profiles of it, and from Asan and Sanver (2002) and Woeginger (2003), who allowed multiple societies and multiple profiles of them.

5. Proofs

Proof of Theorem 1. Suppose that f satisfies **UC** and **IISo** and also that $S_1 \neq S_2$. We show that $S_1 \not\approx_f S_2$. Let p_G^1 be a profile with the property that $p_{v_i}^1 = 0$ for all $v_i \in S_1 \cup S_2$. Since f satisfies **UC**, we get $f(p_{S_1}^1) = f(p_{S_2}^1) = 0$. By $S_1 \neq S_2$ there is some v_j such that $v_j \in S_1$, but $v_j \notin S_2$ (or, equivalently: there is some v_j such that $v_j \in S_2$, but $v_j \notin S_1$). Now let p_G^2 differ from p_G^1 only in that $p_{v_j}^2 \neq 0$. We have again $f(p_{S_2}^2) = 0$ by **UC**. On the other hand, $f(p_{S_1-\{v_j\}}^2) = 0$ by **UC** and so $f(p_{S_1}^2)$ $= p_{v_j}^2 \neq 0$ by **IISo** – which contradicts $S_1 \approx_f S_2$. The proof that $S_1 \approx_f S_2$ if $S_1 = S_2$ is trivial.

Proof of Lemma 1. Let $S = \{v_1, v_2, \dots v_n\}$. Write s for the number of members of S with the property that $p_{v_j} = 1$; m for the number of members of S with the property that $p_{v_j} = -1$, and z for the number of members of S with the property that $p_{v_j} = 0$. We have $s + m + z = n$.

For part (a), we have by definition that $\mu(p_S) = \text{sgn}(\sum_{k=1}^{n} p_{v_k})$. If $\mu(p_S) = 1$, it follows that $\sum_{k=1}^{n} p_{v_k} = \sum_{\substack{k=1 \\ k\neq j}}^{n} p_{v_k} + p_{v_j} \geq 1$. Note that we can have $\mu(p_{S^{-j}}) < 0$ only if $\text{sgn}(\sum_{\substack{k=1 \\ k\neq j}}^{n} p_{v_k}) = -1$, i.e. $\sum_{\substack{k=1 \\ k\neq j}}^{n} p_{v_k}$ < 0; but in this case $\sum_{\substack{k=1 \\ k\neq j}}^{n} p_{v_k} + p_{v_j} < 1$ for each value of p_{v_j} – contradiction. So we established that $\mu(p_{S^{-j}}) \geq 0$ for each $j = 1, \dots n$. But we cannot have $\mu(p_{S^{-j}}) = 0$ for each j. To prove this, consider the following cases:

1) $p_{v_j} = 1$ for all j. Then by definition $\mu(p_{S^{-j}}) = 1$ for each j;
2) $p_{v_j} = 0$ for all j. Then by definition $\mu(p_S) = 0$ – contradiction;

3) $p_{v_j} \geq 0$ for all j and $p_{v_i} = 0$ for some i. Then clearly $\mu(p_{S^{-i}}) = 1$;

4) there is some j such that $p_{v_j} < 1$. Observe that $\mu(p_S) = 1$ entails that $\operatorname{sgn}(\sum_{\substack{k=1 \\ k \neq j}}^{n} p_{v_k}) = 1$ and so $\mu(p_{S^{-j}}) = 1$.

For part (b), note first that we have $s = m$ by definition. If $p_{v_j} = 0$, then clearly $\mu(p_{S^{-j}}) = 0$; if $p_{v_j} = 1$, then $\mu(p_{S^{-j}}) = -1$, because in the number of voters v_i in S^{-j} such that $p_{v_i} = 1$ is $s - 1 < m$; and if $p_{v_j} = -1$, then $\mu(p_{S^{-j}}) = 1$, because in the number of voters v_i in S^{-j} such that $p_{v_i} = 1$ is $m - 1 < s$. But the number of S^{-j}'s with the property that $\mu(p_{S^{-j}}) = 1$ must be equal to the number of S^{-j}'s with the property that $\mu(p_{S^{-j}}) = 1$, because $s = m$.

To prove part (c), suppose first that $\mu(p_S) = 1$. Then part (a) gives that that there is no member S^{-j} of Γ such that $\mu(p_{S^{-j}}) = -1$ and there is a member S^{-j} of it such $\mu(p_{S^{-j}}) = 1$. Then $\mu(p_\Gamma) = 1$. Second, suppose that $\mu(p_S) = 0$. Part (b) gives that Γ has an equal number of S^{-j}'s such that $\mu(p_{S^{-j}}) = 1$ and $\mu(p_{S^{-j}}) = -1$. Then we also have $\mu(p_\Gamma) = 0$.

Finally, for part (d) suppose first again that $\mu(p_S) = 1$. Clearly, $\mu(p_{\Gamma^{-j}}) \geq 0$, because $\mu(p_{S^{-i}}) \geq 0$ for each member of Γ^{-j}. Part one guarantees that there is some S^{-i} with the property that $\mu(p_{S^{-i}}) = 1$. If it is S^{-j}, then $\mu(p_{\{\Gamma^{-j}, S^{-j}\}}) = 1$. If it is one of the members of Γ^{-j}, then $\mu(p_{\Gamma^{-j}})$ and so $\mu(p_{\{\Gamma^{-j}, S^{-j}\}}) = 1$ because $\mu(p_{S^{-j}}) \geq 0$. Secondly, let $\mu(p_S) = 0$. If $\mu(p_{S^{-j}}) = 0$, then by part (b) Γ^{-j} has an equal number of S^{-i}'s such that $\mu(p_{S^{-i}}) = 1$ and $\mu(p_{S^{-i}}) = -1$; therefore $\mu(p_{\Gamma^{-j}}) = 0$ and consequently $\mu(p_{\{\Gamma^{-j}, S^{-j}\}}) = 0$. If $\mu(p_{S^{-j}}) = 1$, then by part (b) the number of S^{-i}'s in Γ^{-j} such that $\mu(p_{S^{-i}}) = -1$ is larger than number of S^{-i}'s in Γ^{-j} such that $\mu(p_{S^{-i}}) = 1$; therefore $\mu(p_{\Gamma^{-j}}) = -1$. So $\mu(p_{\{\Gamma^{-j}, S^{-j}\}}) = \mu(-1, 1) = 0 = \mu(p_S)$ as required.[14]

Proof that α can be defined in terms of μ on domains including ideal votes. Let $S = \{v_1, v_2, \ldots v_n\}$. To show this, we first construct two societies S_1 and S_2:

$S_1 = \{\{v_1, \mathbf{1}^-\}, \{v_2, \mathbf{1}^-\}, \ldots \{v_n, \mathbf{1}^-\}, \{\{v_1, \mathbf{1}^-\}, \mathbf{1}^+\}, \{\{v_2, \mathbf{1}^-\}, \mathbf{1}^+\}, \ldots \{\{v_2, \mathbf{1}^-\}, \mathbf{1}^+\}\}$

$S_2 = \{\{v_1, \mathbf{1}^+\}, \{v_2, \mathbf{1}^+\}, \ldots \{v_n, \mathbf{1}^+\}, \{\{v_1, \mathbf{1}^+\}, \mathbf{1}^-\}, \{\{v_2, \mathbf{1}^+\}, \mathbf{1}^-\}, \ldots \{\{v_2, \mathbf{1}^+\}, \mathbf{1}^-\}\}$

[14] We can prove dual propositions in a similar way, when values 1 and -1 are mutually replaced.

Let s members of $S= \{v_1, v_2, \dots v_n\}$ have $p_{v_j} = 1$, m members have $p_{v_j} = -1$ and z members have $p_{v_j} = 0$, with $s + m + z = n$. Observe that when applied to S_1 the function μ gives s times 0 and $z + m$ times -1; and also s times 1 and $z + m$ times 0. So $\mu(p_{S_1}) = 1$ if $s > z + m$ and $\mu(p_{S_1}) < 1$ in all the other cases. But $s > z + m$ and $s + z + m = n$ entail that $s > n/2$. Analogously, we have that $\mu(p_{S_2}) = -1$ when $m > n/2$ and $\mu(p_{S_2}) > -1$ in all the other cases. Secondly, construct two other societies:

$S_{11} = \{\mathbf{1}^+, \{\mathbf{1}^-, S_1\}\}$

$S_{21} = \{\mathbf{1}^-, \{\mathbf{1}^+, S_2\}\}$.

Clearly, $\mu(p_{S_{11}}) = 1$ if $s > n/2$ and $\mu(p_{S_{11}}) = 0$ in all the other cases; and $\mu(p_{S_{21}}) = -1$ if $m > n/2$ and $\mu(p_{S_{21}}) = 0$ in all the other cases. Finally, put

$\sigma_S = \{S_{11}, S_{21}\}$.

We can easily verify that $\alpha(p_S) = \mu(p_{\sigma_S})$.

Proof of Theorem 3. The proof is by induction on the number of members of S. Suppose first that $n = 2$. Let $S = \{v_1, v_2\}$. As already noted, in this care $\kappa(p_S) = \mu(p_S)$ for all profiles p of S. Now let $n = 3$, i.e. $S = \{v_1, v_2, v_3\}$. Clearly, there are profiles at which κ and μ do not coincide. But consider the society $\sigma_S = \{S^{-1}, S^{-2}, S^{-3}\} = \{\{v_2, v_3\}, \{v_1, v_3\}, \{v_1, v_2\}\}$. We show that:

$$\mu(p_S) = \kappa(p_{\sigma_S}) \text{ for all } p \text{ of } G.$$

This means that we can get $\mu(p_S)$ by the following procedure: we first calculate the value of κ for the three groups, each consisting in two members: $\kappa(p_{\{v_1,v_2\}})$, $\kappa(p_{\{v_2,v_3\}})$ and $\kappa(p_{\{v_1,v_3\}})$. Then we calculate the value of κ for the group consisting in these three groups. To show that the value we get by iteratively applying κ is the same as $\mu(p_S)$ the only interesting cases are when: 1) for two of the members of S it is the case that $p_{v_j} \geq 0$ and also $p_{v_j} = 1$ for at least one of these members, and for the remaining one it is the case that $p_{v_j} = -1$, and 2) the symmetrical ones when -1 is replaced by 1. Note that the construction of the society σ_S makes the order of the members of S irrelevant.

a) If $p_{v_1} = p_{v_2} = 1$ and $p_{v_3} = -1$, then we get $\kappa(p_{\{v_1,v_2\}}) = 1$, $\kappa(p_{\{v_2,v_3\}}) = 0$ and $\kappa(p_{\{v_1,v_3\}}) = 0$, so $\kappa(p_{\sigma_S}) = 1$, in agreement with $\mu(p_S)$.

b) If p_{v_1} = 1, p_{v_2} = 0 and p_{v_3} = -1, then we get $\kappa(p_{\{v_1,v_2\}})$ = 1, $\kappa(p_{\{v_2,v_3\}})$ = -1 and $\kappa(p_{\{v_1,v_3\}})$ = 0, so $\kappa(p_{\sigma_S})$ = 0, again in agreement with $\mu(p_S)$.

(The symmetrical cases can be dealt with in an analogous way.)

Finally let $n > 3$. By induction it holds for each of the n societies $S^{-j} = S - \{v_j\}$ that there is some society $\sigma_{S^{-j}}$ with the property that for all profiles $p_{S^{-i}}$ of S^{-j} it holds that $\mu(p_{S^{-i}}) = \kappa(p_{\sigma_{S^{-j}}})$. We need to show that there is some society σ_S with the property that $\mu(p_S) = \kappa(p_{\sigma_S})$ for all profiles p_S of S. Put $\sigma_S = \{\sigma_{S^{-1}}, \sigma_{S^{-2}}, \dots \sigma_{S^{-n}}\}$.[15]

The proof is trivial for all the cases when $p_{v_i} \geq 0$ for all $v_i \in S$ or $p_{v_i} \leq 0$ for all $v_i \in S$. So suppose that p_{v_i} = 1 for some $v_i \in S$ and p_{v_i} = -1 for some $v_i \in S$. I shall give the proof for the cases when $|\{v_i \in S\colon p_{v_i} = 1\}| \geq |\{v_i \in S\colon p_{v_i} = -1\}|$, i.e. when $\mu(p_S) \geq 0$. The symmetrical cases when $|\{v_i \in S\colon p_{v_i} = 1\}| \leq |\{v_i \in S\colon p_{v_i} = -1\}|$ can be proved in an analogous way.

There are two possibilities. First, let $\mu(p_S) = 0$. By lemma 1b It follows that $|\{v_i \in S\colon p_{v_i} = 1\}| = |\{v_i \in S\colon p_{v_i} = -1\}| = m \geq 0$, where $2m \leq n$. If $m = 0$, then for all v_i it holds that p_{v_i} = 0 and so $\kappa(p_{\sigma_S})$ = 0 by the definition of κ. If $m > 0$, then: a) since there is some v_j such that p_{v_j} = 1 it follows that $\mu(p_{S^{-j}}) = -1$ because $|\{v_i \in S^{-j}\colon p_{v_i} = -1\}| > |\{v_i \in S^{-j}\colon p_{v_i} = 1\}|$; b) since there is some v_j such that p_{v_j} = -1 it follows that $\mu(p_{S^{-j}}) = 1$ because $|\{v_i \in S^{-j}\colon p_{v_i} = 1\}| > |\{v_i \in S^{-j}\colon p_{v_i} = -1\}|$. By induction, for each S^{-j} we have $\mu(p_{S^{-j}}) = \kappa(p_{\sigma_{S^{-j}}})$, so we have $\kappa(p_{\sigma_{S^{-j}}}) = 1$ in some cases and $\kappa(p_{\sigma_{S^{-j}}}) = -1$ in other cases, whence by the definition of κ we get $\kappa(p_{\sigma_S}) = 0$, whence $\kappa(p_{\sigma_S}) = \mu(p_S)$.

Second, let $\mu(p_S) = 1$. It follows that $|\{v_i \in S\colon p_{v_i} = 1\}| = m_1 > |\{v_i \in S\colon p_{v_i} = -1\}| = m_2$. Let v_j be some member of S. As shown in the proof of lemma 1a we have: i) if p_{v_j} = -1, then $\mu(p_{S^{-j}}) = 1$, since clearly $|\{v_i \in S^{-j}\colon p_{v_i} = 1\}| > |\{v_i \in S^{-j}\colon p_{v_i} = -1\}|$; consequently, by induction $\kappa(p_{\sigma_{S^{-j}}}) = 1$. Similarly, observe that: ii) if p_{v_j} = 0, then $\mu(p_{S^{-j}}) = 1$ and so by induction $\kappa(p_{\sigma_{S^{-j}}})$ =

[15] For example, let $S = \{v_1, v_2, v_3, v_4\}$. Since all S^{-j} have three members, we can apply the above procedure and get: $\sigma_{S^{-1}} = \{\{v_4, v_2\}, \{v_2, v_3\}, \{v_4, v_3\}\}$, $\sigma_{S^{-2}} = \{\{v_1, v_4\}, \{v_4, v_3\}, \{v_1, v_3\}\}$, $\sigma_{S^{-3}} = \{\{v_1, v_2\}, \{v_2, v_4\}, \{v_1, v_4\}\}$and $\sigma_{S^{-4}} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$. Therefore:

$\sigma_S = \{\{\{v_4, v_2\}, \{v_2, v_3\}, \{v_4, v_3\}\}, \{\{v_1, v_4\}, \{v_4, v_3\}, \{v_1, v_3\}\}, \{\{v_1, v_2\}, \{v_2, v_4\}, \{v_1, v_4\}\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}\}$

1. Finally, iii) if p_{v_j} = 1, then we must have μ($p_{S^{-j}}$) ≥ 0 and thus κ($p_{\sigma_{S^{-j}}}$) ≥ 0. There are two subcases:

a) If $m_2 > 0$ (i.e. there is at least a v_j with the property that $p_{\sigma_{S^{-j}}}$ = -1), then there is some S^{-j} such that μ($p_{S^{-j}}$) = 1, which entails κ($p_{\sigma_{S^{-j}}}$) = 1, and so κ(p_{σ_S}) = 1 = μ(p_S).

If $m_2 = 0$ (i.e. there is no v_j with the property that p_{v_j} = -1), it follows that there is some $v_j \in S$ such that p_{v_j} = 1 and for all the other $v_j \in S$ it holds that $p_{v_j} \geq 0$. If p_{v_j} = 1 for all $v_j \in S$, then given that $n > 3$, we must have $m_1 - 1 > 0$ and thus μ($p_{S^{-j}}$) = 1 for all S^{-j}. Consequently, κ($p_{\sigma_{S^{-j}}}$) = 1, and so κ(p_{σ_S}) = 1 = μ(p_S). But if p_{v_j} = 0 for some $v_j \in S$, then by induction (see the argument in the second case above) μ($p_{S^{-j}}$) = 1. We can again conclude that κ($p_{\sigma_{S^{-j}}}$) = 1, whence κ(p_{σ_S}) = 1 = μ(p_S).

Remark on theorem 4: if **Neu** and **Mon** hold, then **NZ** entails **SP**. Suppose that f satisfies **NZ**. Then if at a profile p_S we have that $f(p_S) = 0$, there must be a zigzag sequence p_S^1, p_S^2, … p_S^m of profiles of S such that f(p_S^k) = 1 for some profile p_S^k in this sequence (this is so because f satisfies **Neu**). Then by **Mon** we get that f($p_S^{'}$) = 1 at a profile $p_S^{'}$ where $p_{v_j} \geq 0$ for all $v_j \in S$ and p_{v_j} = 1 for some $v_j \in S$.

Proof of Theorem 5. We need to show that for each S there is some (higher-order) binary society σ_S such that $f^n(p_S) = f^2$(p_{σ_S}). The society σ_S is recursively constructed as follows:

i) if $S = \{i_1, i_2, i_3\}$, then $\sigma_S = \{\{i_1, i_2\}, \{i_3\}\}$;

ii) if $S = \{i_1, \ldots i_n\}$, then $\sigma_S = \{ \sigma_{S-\{i_n\}}, \{i_n\}\}$.

We can check that since the swfs Min^2, Max^2 and υ^2 are associative, they can all be extended to the n-ary case. Consider the unanimity rule υ. We can easily see that it satisfies associativity: υ^n($p_{\{i_1,i_2,\ldots i_{n-1},i_n\}}$) = υ^2(υ^{n-1}($p_{\{i_1,i_2,\ldots i_{n-1}\}}$), p_{i_n}). Suppose that υ^n($p_{\{i_1,i_2,\ldots i_{n-1},i_n\}}$) = 1. By definition, this means that p_{i_k} = 1 for all k. But then υ^{n-1}($p_{\{i_1,i_2,\ldots i_{n-1}\}}$) = 1 and also p_{i_n} = 1. Therefore υ^2(υ^{n-1}($p_{\{i_1,i_2,\ldots i_{n-1}\}}$), p_{i_n}) = υ^2(1, 1) = 1. The case when υ^n($p_{\{i_1,i_2,\ldots i_{n-1},i_n\}}$) = -1 is similar. If υ^n($p_{\{i_1,i_2,\ldots i_{n-1},i_n\}}$) = 0, we must have at least two individuals i_k and $i_{k'}$ such that $p_{i_k} \neq p_{i_{k'}}$. If both are in $\{i_1, \ldots i_{n-1}\}$, then $\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}) = 0$ and so $\upsilon^{n}(p_{\{i_1,i_2,...i_{n-1},i_n\}}) = \upsilon^{2}(\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}), p_{i_n}) = \upsilon^{2}(\upsilon^{n-1}(0, p_{i_n}) = 0$. If one of them, e.g. $i_{k'}$, is p_{i_n}, then we have three cases:

1) $p_{i_n} = 1$. Then if $p_{i_k} = -1$ we may have either $\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}) = -1$ or $\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}) = 0$. But in both subcases $\upsilon^{2}(\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}), p_{i_n}) = 0$. If $p_{i_k} = 0$, then $\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}) = 0$ and again $\upsilon^{2}(\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}), p_{i_n}) = \upsilon^{2}(0, 1) = 0$.
2) $p_{i_n} = -1$. This case is analogous to the first case.
3) $p_{i_n} = 0$. Then we always get $\upsilon^{2}(\upsilon^{n-1}(p_{\{i_1,i_2,...i_{n-1}\}}), 0) = 0$.

Proof of Theorem 6. I shall start with α. It suffices to show that α^2 cannot be extended to α^3. So, let $S = \{i_1, i_2, i_3\}$. We prove that there is no higher order society σ_S such that for all profiles S it holds that $\alpha^3(p_S) = \alpha^2(p_{\sigma_S})$ and σ_S is binary. Observe that $B(\sigma_S) = B(S)$, i.e. the only individuals that may occur in σ_S are i_1, i_2 and i_3. Suppose to the contrary that there is such a society σ_S and let a profile p_S^1 of it be given by: $p_{i_1} = 0$; $p_{i_2} = 1$; $p_{i_3} = 1$. By the definition of α we must have $\alpha^3(p_S) = 1$. Now suppose that there is some society S' nested in σ_S in which there is an occurrence of i_1. Since $p_{i_1} = 0$ we get $\alpha^2(p_{S'}) = 0$ and by iterating the applications of α^2 it follows that $\alpha^2(p_{\sigma_S}) = 0$ – contradiction. So σ_S cannot contain any occurrence of i_1. Similarly, if we take into account the profile p_S^2 of σ_S defined by: $p_{i_1} = 1$; $p_{i_2} = 0$; $p_{i_3} = 1$ we must conclude that i_2 does not occur in σ_S. Finally, focusing on the profile p_S^3 of σ_S defined by: $p_{i_1} = 1$; $p_{i_2} = 1$; $p_{i_3} = 0$ we get that i_3 does not occur in σ_S. It follows that σ_S can only be the empty set ∅. But by definition for any profile of ∅ we have $\alpha(p_\varnothing) = 0$ – contradiction. Therefore, no σ_S satisfies the property that $\alpha^3(p_S) = \alpha^2(p_{\sigma_S})$ for all profiles p_S.

Moving to κ, it is again sufficient to show that κ^3 is not definable in terms of κ^2. Since $\kappa^2 = \mu^2$, we need to show that for any $S = \{v_1, v_2, v_3\}$ there is no binary society σ_S such that $\kappa^3(p_S) = \kappa^2(p_{\sigma_S})$ for all profiles p_S of S. The proof has two steps.

Step 1. We show that function κ^2 is neutral on the domain of higher order societies. $= - p_{v_2}^2$. Clearly, we have $\kappa^2(p_{\{v_1,v_2\}}^1) = -\kappa^2(p_{\{v_1,v_2\}}^2)$. Now suppose that $\sigma_\kappa = \{\sigma_\lambda^1, \sigma_\lambda^2\}$ where both σ_λ^1 and σ_λ^2 are binary societies. By induction, if $p_{v_1}^1 = - p_{v_1}^2$ and $p_{v_2}^1 = - p_{v_2}^2$ and $p_{v_3}^1 = - p_{v_3}^2$ then $\kappa^2(p_{\sigma_\lambda^1}^1) = -$

$\kappa^2(p^2_{\sigma^1_\lambda})$ and $\kappa^2(p^1_{\sigma^2_\lambda})$= -$\kappa^2(p^2_{\sigma^2_\lambda})$. Then $\kappa^2(p^2_{\sigma_\lambda}) = \kappa^2(\kappa^2(p^2_{\sigma^1_\lambda}), \kappa^2(p^2_{\sigma^2_\lambda})) = \kappa^2(-\kappa^2(p^1_{\sigma^1_\lambda}), -\kappa^2(p^1_{\sigma^2_\lambda}))$ = $-\kappa^2(\kappa^2(p^1_{\sigma^1_\lambda}), \kappa^2(p^1_{\sigma^2_\lambda})) = -\kappa^2(p^1_{\sigma_\lambda})$.

Step 2. Suppose that at profile p^1_S we have $p^1_{v_1} = 1$ and $p^1_{v_2} = 1$ and $p^1_{v_3} = -1$. By definition, $\kappa^3(p^1_S) = 0$. Consider also three other profiles defined as follows[16]:

1) at p^2_S we have that $p^2_{v_1} = 1$ and $p^2_{v_2} = -1$ and $p^2_{v_3} = -1$; we get $\kappa^3(p^2_S) = 0$.
2) at p^3_S we have that $p^3_{v_1} = 1$ and $p^3_{v_2} = -1$ and $p^3_{v_3} = 1$; we get $\kappa^3(p^3_S) = 0$.
3) at p^4_S we have that $p^4_{v_1} = -1$ and $p^4_{v_2} = -1$ and $p^4_{v_3} = 1$; we get $\kappa^3(p^4_S) = 0$.

Observe that $p^1_S > p^2_S < p^3_S > p^4_S = - p^1_S$. On the other hand, if κ^3 is definable in terms of κ^2, then there must be some society $\sigma_S = \{\sigma^1_S, \sigma^2_S\}$ where both σ^1_S and σ^2_S are binary societies and $\kappa^3(p_S) = \kappa^2(p_{\sigma_S})$ for all profiles p_S. The idea of the proof is to show that these conditions imply a contradiction.

First, suppose that both σ^1_S and σ^2_S are binary first-order societies, i.e. they are subsets of *S*. Since in each of the four profiles two voters have 1 and one voter has -1, it follows that any combination of them to form binary societies must give 1 for one combination and 0 for the other, so $\kappa^2(p_{\sigma_S}) = \kappa^2(\kappa^2(p_{\sigma^1_S}), \kappa^2(p_{\sigma^2_S})) = \kappa^2(1, 0) = 1 \neq \kappa^3(p_S) = 0$ – contradiction. Second, suppose by induction that σ^1_S and σ^2_S are binary higher-order societies. Since κ^2 is monotonic, we must have for each profile $p^k_{\sigma_S}$ (k = 1, …4) that[17]: i) $\kappa^2(p^k_{\sigma^1_S}) \leq \kappa^2(p^{k+1}_{\sigma^1_S})$ and $\kappa^2(p^k_{\sigma^2_S}) \leq \kappa^2(p^{k+1}_{\sigma^2_S})$, or ii) $\kappa^2(p^k_{\sigma^1_S}) \geq \kappa^2(p^{k+1}_{\sigma^1_S})$ and $\kappa^2(p^k_{\sigma^2_S}) \geq \kappa^2(p^{k+1}_{\sigma^2_S})$. On the other hand, $\kappa^3(p^k_S) = \kappa^3(p^{k+1}_S) = 0$. So $\kappa^2(p^k_{\sigma^1_S}) = \kappa^2(p^{k+1}_{\sigma^1_S})$ and $\kappa^2(p^k_{\sigma^2_S}) = \kappa^2(p^{k+1}_{\sigma^2_S})$. Therefore, for each *k* we get:

$$\kappa^2(p^k_{\sigma^1_S}) = \kappa^2(p^1_{\sigma^1_S}) \text{ and } \kappa^2(p^k_{\sigma^2_S}) = \kappa^2(p^1_{\sigma^2_S}).$$

But by induction we have $\kappa^2(p^k_{\sigma^1_S}) \neq 0$ or at least $\kappa^2(p^k_{\sigma^2_S}) \neq 0$. Suppose for example that $\kappa^2(p^k_{\sigma^1_S}) \neq 0$. Then we get $\kappa^2(p^1_{\sigma^1_S}) = \kappa^2(p^4_{\sigma^1_S}) \neq 0$. However, we noticed already that $p^1_S = - p^1_S$, which contradicts the neutrality of κ.

[16] This argument is inspired by the necessity part of Fine's (1972) proof of his theorem 3.
[17] If $k = 4$, we put $k + 1 = 1$.

Proof of Theorem 7. We show that for each society $S \subseteq G$ formed of n members there is some (higher-order) society σ_S such that 1) σ_S is binary; and 2) for all profiles p_S of S we have: $\mu^n(p_S) = \mu^2(p_{\sigma_S})$. Given that the profile p_G is kept constant in the proof, whenever possible we shall omit references to it. The proof of the theorem is by induction on the number of members of S. For $n = 2$, the proof is trivial. For $n = 3$, let $S^{-1} = \{v_2, v_3\}$, $S^{-2} = \{v_1, v_3\}$, $S^{-3} = \{v_1, v_2\}$. Further, put $\Gamma^{-3} = \{S^{-1}, S^{-2}\} = \{\{v_2, v_3\}, \{v_1, v_3\}\}$. Finally, let $\sigma_S = \{\Gamma^{-3}, S^{-3}\} = \{\{\{v_2, v_3\}, \{v_1, v_3\}\}, \{v_1, v_2\}\}$. Since σ_S is clearly binary, the function μ^2 can be iteratively applied to it. Lemma 1d gives $\mu^2(p_{\sigma_S}) = \mu(p_{\{\Gamma^{-j}, S^{-j}\}}) = \mu^3(p_S)$.

Now consider the case when S contains n members. The society σ_S is recursively constructed as follows: a) if $S = \{v_i, v_j\}$, then $\sigma_S = S$; b) if $S = \{v_1, v_2, \dots v_n\}$, then: 1) construct the n societies $S^{-j} = S - \{v_j\}$; 2) construct n societies $\sigma_{S^{-1}}, \sigma_{S^{-2}}, \dots \sigma_{S^{-n}}$ with the property that $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}})$ for each j; 3) construct a new society $\Gamma^{-n} = \{\sigma_{S^{-1}}, \sigma_{S^{-2}}, \dots \sigma_{S^{-(n-1)}}\}$; 3) finally, construct the society $\sigma_S = \{\sigma_{\Gamma^{-n}}, \sigma_{S^{-n}}\}$. Note that since S has n members, all S^{-k}'s and Γ^{-n} have n -1 members. Clearly, each $\sigma_{S^{-k}}$ and $\sigma_{\Gamma^{-n}}$, as well as σ_S are binary. (Observe also that the definition of σ_S also covers the case when $n = 3$.)

By induction we have that for each society S^{-j} ($j = 1, 2, \dots n$) there is some society $\sigma_{S^{-j}}$ with the property that $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}})$. I shall prove that $\mu^n(p_S) = \mu^2(p_{\sigma_S})$ holds for all profiles of S.

Case 1: $\mu^n(p_S) = 1$. By the lemma 1a[18] all S^{-j}'s are such that $\mu^{n-1}(p_{S^{-j}}) \geq 0$ and there is some j such that $\mu^{n-1}(p_{S^{-j}}) = 1$. By induction, since S^{-j} contains exactly n - 1 members, we have that $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}})$. Moving to $\Gamma^{-n} = \{\sigma_{S^{-1}}, \sigma_{S^{-2}}, \dots \sigma_{S^{-(n-1)}}\}$, observe first that this society has $n - 1$ members and so $\mu^{n-1}(p_{\Gamma^{-n}}) = \mu^2(p_{\sigma_{\Gamma^{-n}}})$. Second, for each $j < n$ we have: $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}})$. Now we apply to Γ^{-n} the argument given above for the societies S^{-j}. We have two subcases:

i) $\mu^{n-1}(p_{S^{-n}}) = 0$. Then we must have $\mu^{n-1}(p_{S^{-j}}) = 1 = \mu^2(p_{\sigma_{S^{-j}}})$ for some $j < n$, while for the all the other $k < n$ we have $\mu^{n-1}(p_{S^{-k}}) = \mu^2(p_{\sigma_{S^{-k}}}) \geq 0$, which entails that $\mu^2(p_{\sigma_{\Gamma^{-n}}}) = 1$ and thus $\mu^2(p_{\sigma_{\{\Gamma^{-n}, \sigma_{S^{-n}}\}}}) = \mu^2(1,0) = 1 = \mu^n(p_S)$;

[18] Similarly, we can prove dual propositions, when value -1 replaces 1. This result is necessary in the proof of Case 2.

ii) $\mu^{n-1}(p_{S^{-n}}) = 1$. Then $\mu^2(p_{\sigma_{\Gamma^{-n}}}) = a \geq 0$, which gives again $\mu^2(a,1) = 1 = \mu^n(p_S)$.

Case 2: $\mu^n(p_S) = -1$. The proof is just like in case 1.

Case 3: $\mu^n(p_S) = 0$. Let s be again the number of v_j's such that $p_{v_j} = 1$; m the number of v_j's such that $p_{v_j} = -1$; and z the number of v_j's such that $p_{v_j} = 0$. By the definition of μ we must have that $m = s$. Keeping in mind that $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}})$ for all j, the value of $\mu^2(p_{\sigma_{S^{-j}}})$ is determined as follows:

i) if $p_{v_j} = 0$, then clearly at S^{-j} we still have $m = s$, because p_{v_j} is a z, and so $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}}) = 0$;

ii) if $p_{v_j} = 1$, then we have $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}}) = -1$, because at S^{-j} we have $m > s - 1$.

iii) if $p_{v_j} = -1$, then analogously we get $\mu^{n-1}(p_{S^{-j}}) = \mu^2(p_{\sigma_{S^{-j}}}) = 1$.

So there are s societies S^{-j} such that $\mu^{n-1}(p_{S^{-j}}) = -1$ and m societies S^{-j} such that $\mu^{n-1}(p_{S^{-j}}) = 1$. Again, $m = s$. Further, the society Γ^{-n} contains exactly n-1 members, and so by induction $\mu^{n-1}(p_{\Gamma^{-n}}) = \mu^2(p_{\sigma_{\Gamma^{-n}}})$. Moreover, the society $\sigma_S = \{ p_{\sigma_{\Gamma^{-n}}}, \sigma_{S^{-n}} \}$ contains exactly two members and therefore we can apply μ^2 to it. There are three possibilities:

i) $\mu^{n-1}(p_{S^{-j}}) = 1$. Then in Γ^{-n} the number of S^{-j}'s such that $\mu^{n-1}(p_{S^{-j}}) = -1$ is larger than the number of S^{-j}'s such that $\mu^{n-1}(p_{S^{-j}}) = 1$, which entails that $\mu^{n-1}(p_{\Gamma^{-n}}) = -1 = \mu^2(p_{\sigma_{\Gamma^{-n}}})$. Then $\mu^2(p_{\sigma_{\{\Gamma^{-n}, \sigma_{S^{-n}}\}}}) = \mu^2(-1, 1) = 0 = \mu^n(p_S)$.

ii) $\mu^{n-1}(p_{S^{-j}}) = -1$. This case is similar to (i).

iii) $\mu^{n-1}(p_{S^{-n}}) = 0$. Then in Γ^{-n} the number of S^{-j}'s such that $\mu^{n-1}(p_{S^{-j}}) = -1$ is equal to the number of S^{-j}'s such that $\mu^{n-1}(p_{S^{-j}}) = 1$. Therefore $\mu^{n-1}(p_{\Gamma^{-n}}) = 0 = \mu^2(p_{\sigma_{\Gamma^{-n}}})$ and so $\mu^2(p_{\sigma_{\{\Gamma^{-n}, \sigma_{S^{-n}}\}}}) = \mu^2(0, 0) = 0 = \mu^n(p_S)$. ■

Proof of Theorem 8. Notice first that μ^2 satisfies the four axioms. Conversely, we need to show that if a swf f satisfies these axioms, then it must be exactly μ^2. If $|S| = 1$, i.e. it is a singleton

$\{v_1\}$, then by **F** we have $f^1(p_{\{v_1\}}) = p_{v_1} = \mu^1(p_{\{v_1\}})$. So let $|S| = 2$, where $S = \{v_1, v_2\}$. We show that $f^2(p_{\{v_1,v_2\}}) = \text{sgn}(p_{v_1} + p_{v_2})$ in all possible cases. We have:

i) for profiles (1, 1) and (-1, -1) axiom **BU** in conjunction with **F** gives immediately $f^2(p_{\{v_1,v_2\}}) = 1$, respectively $f^2(p_{\{v_1,v_2\}}) = -1$;

ii) for profiles (1, -1) and (-1, 1) axioms **F** and **SET** give $f^2(p_{\{v_1,v_2\}}) = 0$;

iii) for profiles (-1, 0) and (0, -1) axioms **F** and **IIS** give $f^2(p_{\{v_1,v_2\}}) = -1$;

iv) for profiles (1, 0) and (0, 1) axioms **F** and **IIS** give $f^2(p_{\{v_1,v_2\}}) = 1$; finally,

v) for the profile (0, 0) axioms **F** and **IIS** give $f^2(p_{\{v_1,v_2\}}) = 0$. ■

6. Conclusions

The main objective of this paper was to show that the study of swfs like the simple majority rule μ on domains including higher-order societies requires a careful examination of the properties they retain and also brings about new topics to be addressed. I argued that the reducibility of a swf to another is a powerful notion that gives us new insights about its properties and its relations with other swfs. I showed that when applied to higher-order domains μ fails to satisfy some of its standard properties (responsiveness, anonymity and capacity to work as an identifying criterion), while other properties of μ come to the fore (for example, the "reducibility to subsocieties"). The theorems I presented in sections 3 and 4 appeal to the notion of reducibility of one swf to another. The old results of Murakami, Fishburn and Fine focused on μ and succeeded to identify the class of swfs that can be reduced to μ (or: expressed in terms of it). An important side-result they proved was that the consensus rule κ cannot be reduced to μ. In this paper I proved that, conversely, κ is able to reduce μ; I also gave a characterization of the class of swfs reducible to κ. It is interesting to observe that the notion of reducibility makes sense not only in the relation between two different swfs, but also in the relation between a swf and its binary part, i.e. its applications to societies formed of only two members. Many swfs, μ included, can be reduced to their binary parts; others do not.

An interesting result was reported in theorem 3b. There two "ideal" voters were introduced: they are characterized by the fact that they vote in the same way in all profiles (either 1 or -1). This procedure has a number of virtues worth highlighting. Consider the ideal voter $\mathbf{1}^+$. It plays the same role as a swf t (top) which at all profiles p_S of S gives the value 1: we have $t(p_S) = 1$. Similarly, the ideal voter $\mathbf{1}^-$ plays the same role as a swf b (bottom) which gives $b(p_S) = -1$ at all profiles p_S.

So, theorem 3b can be rephrased as follows: the class of swfs reducible to the set formed of three swfs: the simple majority rule μ, the top function *t* and the bottom function *b* is exactly $\Phi_{\{\mathbf{Mon}\}}$. A future interesting question would be to try to determine the classes of swfs reducible to other sets of swfs.

References

Alcantud, J.C.R. (2019). Yet another characterization of the majority rule, *Economics Letters*, 177, 52 – 55.

Asan, G., Sanver, M.R., 2002. Another characterization of the majority rule. Economics Letters 75, 409–413.

Batra, R., Pattanaik, P.K., 1972. Transitive Multi-Stage Majority Decisions with Quasi-Transitive Individual Preferences. Econometrica 40, 1121-1135.

Buchanan, J. M., Tullock, G. 1999. The Calculus of Consent. Logical Foundations of Constitutional Democracy. In *The Collected Works of James M. Buchanan*, volume 3. Indianapolis: Liberty Fund.

Campbell, D.E. and J.S.Kelly (2000) "A simple characterization of majority rule" *Economic Theory* **15**, 689 – 700

Fine, K., 1972. Some Necessary and Sufficient Conditions for Representative Decision on Two Alternatives. Econometrica 40, 1083 – 1090.

Fishburn, P.C. (1971). The Theory of Representative Majority Decision. Econometrica 39, 273 – 284.

Hall, U. (1964). Voting Procedure in Roman Assemblies, *Historia: Zeitschrift für Alte Geschichte*, 13, 3, pp. 267 – 306.

Locke, J. (2003). *Two Treatises of Government and A Letter Concerning Toleration*, Yale University Press, New Haven and London.

May, K.O. (1952). "A set of independent necessary and sufficient conditions for simple majority decisions" *Econometrica* **20**, 680–684

Miroiu, A., Partenie, C. (2019). Collective Choice in Aristotle, *Constitutional Political Economy*, 30, 261 –281.

Mommsen, Th. (1894). *The History of Rome*, vol. III, Richard Bentley & Son, London.

Murakami, Y., 1966. Formal Structure of Majority Decisions. *Econometrica* 34, 709 – 718.

Murakami, Y., 1968. Logic and Social Choice. Dover, New York.

Quesada, A. (2013a). To Majority through the Search for Unanimity, *Journal of Public Economic Theory*, 15, 5, pp. 729 – 735.

Quesada, A. (2013b). The majority rule with a chairman, *Social Choice and Welfare*, 40, 679 – 691.

Rousseau, J.-J. (2002). *The Social Contract and The First and Second Discourses*, Yale University Press, New Haven and London.

Rousseau, J.-J. (2005). Considerations on the Government of Poland and on Its Planned Reformation, in *The Collected Writings of Rousseau*, vol II, University Press of New England, Lebanon, NH.

Sanver, M. R. (2009). Characterizations of majoritarianism: a unified approach, *Social Choice and Welfare*, 33, 159 – 171,

Sengupta, M., 1974. On a Concept of Representative Democracy. Theory Decision 5, 249 – 262.

Taylor, L. S. (1966). *Roman voting assemblies: From the Hannibalic War to the dictatorship of Caesar*, The University of Michigan Press, Ann Arbor.

Woeginger, G.J. (2003) A new characterization of the majority rule *Economics Letters* 81, 89–94.

Xu, Y. and Z. Zhong (2010) Single profile of preferences with variable societies: A characterization of simple majority rule *Economics Letters* 107, 119-121