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Single-profile axiomatizations of the plurality and the simple majority rules

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Abstract

This note presents a new axiomatization of the plurality rule. The proof appeals to a single-profile approach. An axiomatization of the simple majority rule is obtained as a corollary of it. It is shown that besides axioms that describe properties of these rules, an important role is played by restrictions on the form of the choice rules or on the general frame (the number of available alternatives).

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1. Introduction

Finding properties that uniquely characterize a voting rule helps us better understand both the rule and the way in which it relates to other rules. The majority and plurality rules are among the best known and most used rules in both committee and mass elections. More than six decades ago K. May (1952) proved, in what now many would consider a minor classic paper, that the simple majority rule can be singled out by means of three properties it and only it satisfies: Anonymity, Neutrality and Responsiveness. Such properties have the following form: given a fixed society, they indicate how voters' choices, or changes in the voters' choices, affect the social, or group choice. In general, they require taking into account more than one profile, i.e. more possible distributions of the voters' preferences. Following this multi-profile approach, in the past decades many other axiomatizations of the simple majority rules were proposed: Campbell and Kelly (2000), Woeginger (2003), Sanver (2009), etc. Axiomatization theorems for the plurality rule were proved in Richelson (1978), Roberts (1991), Ching (1996), Goodin and List (2006) and Yeh (2008). These authors also used a multiple-profile approach.

A distinct approach was followed by Xu and Zhong (2010) and Quesada (2011). These authors proved that the majority rule can be axiomatized by appealing only to properties that connect either voters' choices with group choices, or choices of different groups, while keeping fixed the voters' choices. They showed that the appeal to more than one profile is superfluous, and so that a single-profile axiomatization of the simple majority rule can be offered. Wu, Xu, and Zhong (2015) gave a similar single profile axiomatization of the approval voting.

In this paper I present single-profile axiomatizations, in a unified framework, of both the simple majority and the plurality rules. The axiomatization theorem concerns social choice rules: they aggregate individual choices and give, for each profile, a social choice set, i.e. the collection of alternatives the group chooses. This approach is different from other axiomatizations, in which individuals are attached a preference relation, and the aggregating rule produces a social preference relation. The axiomatizations appeal to only two properties: Strong Additive Responsiveness (**SAR**) and Disjoint Individual Set Choices (**DISC**). When voters are only allowed to choose exactly one candidate or be indifferent, an axiomatization of the plurality rule is obtained; when the set of candidates is restricted to only two, we get an axiomatization of the simple majority rule.

The paper is organized as follows. In section 2 I formulate the framework in which the two rules are defined. Section 3 presents the properties I shall use to axiomatize them. A distinction is made between such properties and restrictions on the general framework. Restrictions can affect the agenda or the definition of a social choice functions. I shall also present some auxiliary properties and study their connections. In section 4 the axiomatization theorem is proved. Section 5 concludes.

2. The formal framework

The formal framework of social choice theory is here introduced. A choice profile is a structure $\mathbf{p} = (G, X, c, C)$, where G is the set of voters v_1, v_2, … v_n and X (the agenda) is the set of candidates a_1, a_2, … a_m; c gives the (individual) choice set $c(v_i) \subseteq X$ for each voter v_i; it is assumed that $c(v_i) \neq \varnothing$. C is the social choice function, which gives for each subgroup $S \subseteq G$ a nonempty social choice set $C(S)$. An individual is concerned if $c(v_i) \neq X$. Similarly, a group S is concerned if $C(S) \neq X$. I shall say that an alternative is properly chosen by an individual v_i if v_i is concerned and the alternative belongs to her choice set. For a group $S = \{v_1, v_2, \dots v_k\}$, say that it has individual disjoint choices if: 1) $c(v_i) \in X$ for all $v_i \in S$, i.e. individuals in S have singletons as their choices, and 2) $c(v_i) \neq c(v_j)$ for any two voters v_i and v_j in S.

In this paper I shall give axiomatizations of the simple majority rule μ and the plurality rule π. The axiomatizations appeal to properties of the social choice rules that require a reference to a single profile. In what follows a fixed profile is assumed and therefore I skip an explicit reference to it. The rules μ and π are defined as follows:

The simple majority rule μ: let $X = \{a, b\}$. Then

$\mu_S(X) = \{a\}$ if $|\{v_i: c(v_i) = \{a\}\}| > |\{v_i: c(v_i) = \{b\}\}|$;

$\mu_S(X) = \{b\}$ if $|\{v_i: c(v_i) = \{b\}\}| > |\{v_i: c(v_i) = \{a\}\}|$;

$\mu_S(X) = \{a, b\}$ if $|\{v_i: c(v_i) = \{a\}\}| = |\{v_i: c(v_i) = \{b\}\}|$.

The plurality rule π:

$\pi_S(X) = \{a : |\{v_i: c(v_i) = \{a\}\}| \geq (|\{v_i: c(v_i) = \{b\}\}|, \text{ for all } b \in X)$

By μ, a candidate is socially chosen if she is chosen by more voters than its alternative. If the two candidates get the same number of votes, then the social choice set consists in both of them; in this case the group is unconcerned. By π a candidate is socially chosen if the number of voters who choose only her is greater than or equal to the number of voters who choose any other single candidate (it is of course possible that more than one candidate, possibly all of them, be socially chosen). Note that in the case of the plurality rule we take into account only those voters who choose exactly one candidate. Voters who are unconcerned (i.e. choose all the candidates on the agenda) do not contribute to the social choice.

3. The axioms

The following two axioms will be used to characterize the plurality and the simple majority choice rules:

Strong Additive Responsiveness (**SAR**). If $S \cap T = \emptyset$ and $C(T) \subseteq C(S)$, then $C(S \cup T) = C(T)$.

Disjoint Individual Set Choices (**DISC**). Let $S = \{v_1, v_2, \dots v_m\}$. If S has individual disjoint choices, then $C(S) = \bigcup_{i=1}^{m} c(v_i)$.

Both properties have an intuitive import. **SAR** requires that, if two groups are disjoint, but a group's choice set is included in the choice set of the other group, then the collective choice of the group resulting by combining the two groups is exactly the stricter choice set. By the property of Disjoint Individual Set Choices (**DISC**) if the members of a group have mutually disjoint choice sets, then the group's choice is the union of all these choice sets. For example, if a group is formed of two individuals, and one of them chooses some alternative a, while the other chooses an alternative b, then the choice set of the group consists in these two alternatives.

Now consider the following two restrictions:

Binary agenda (**BA**). $|X| = 2$.

Single choices (**SC**). For all $v_i \in G$, either $c(v_i) \in X$, or $c(v_i) = X$.

BA is a restriction on the admissible profiles: the set of candidates includes only two members. **SC** is a restriction on the form of individual choice functions. It states that a voter can either choose exactly one of the n candidates or be indifferent between all candidates; so she cannot choose, e.g., two or three or n-1 candidates. The approval voting, for example, does not satisfy **SC**. We can immediately see that if **BA** holds, then **SC** is also valid: for in this case an individual can choose one of the two candidates or be indifferent between them.

The main result of this paper is expressed by the following theorem.

Theorem 1.

a) If **SC** holds, then the plurality rule π is the only social choice function that satisfies the properties **SAR** and **DISC**.
b) If **BA** holds, then the simple majority rule μ is the only social choice function that satisfies the properties **SAR** and **DISC**.

Clearly, since, as I mentioned above, **SC** holds if **BA** holds, part (b) of the theorem yields immediately from (a). However, in section 4 I shall spend some space to discuss the form properties **SAR** and **DISC** take in the special case when the agenda consists (under **BA**) in only two candidates.

Consider also two other properties of social choice functions:

Faithfulness (**F**). $C(\{v_i\}) = c(v_i)$.
Unanimity (**U**). If $c(v_i)= Y$ for all $v_i \in S$, then $C(S) = Y$.

F states that a group consisting in only one individual chooses exactly what that individual does (for this property, see also Miroiu: 2004; Xu, Zhong: 2010). **U** requires that if all the members of the group make the same choice, then the group will collectively make that choice.

The following proposition relates the properties I have introduced.

Proposition 1.

1) **DISC** entails **F**.
2) **SAR** and **F** entail **U**.

Proof. Part (1) is the result of taking S as a singleton in the definition of **DISC**. For part (2), let $S = \{v_1, v_2, \dots v_m\}$ and $c(v_i) = Y \subseteq X$ for all $v_i \in S$. By **F**, for each $v_i \in S$ we have that $C(\{v_i\}) = c(v_i)$. Now we apply **SAR** iteratively: first, it is applied to the sets $\{v_1\}$ and $\{v_2\}$. Since $C(\{v_1\})= C(\{v_2\}) = Y$, we get that $C(\{v_1\}\cup\{v_2\}) = Y$. Secondly, **SAR** applies to the sets $\{v_1, v_2\}$ and $\{v_3\}$, and finally to the sets $\{v_1, v_2, \dots v_{m-1}\}$ and $\{v_m\}$ to yield: $C(S) = $ Y. ■

4. Proof of the theorem

With these preparatory results, we can turn to the proof of Theorem 1a. So, let **SC** hold. It is not difficult to show that π satisfies properties **SAR** and **DISC**. Let us assume that a choice function C satisfies the properties **SAR** and **DISC**. Assuming also **SC**, $c(v_j)$ must be either a singleton or X. Clearly, by Proposition 1 C also satisfies **F** and **U**. We want to show that C is exactly π, i.e. that for each group S we have $C(S) = \pi(S)$.

In the case when all the members of S are unconcernd, i.e. $c(v_i) = X$ for all v_i, all alternatives receive the same number of votes and so by the definition of π it holds that $\pi(S) = X$. On the other hand, given that C satisfies **U**, we also get $C(S) = X$.

So let us suppose that at least one member v_i of S is concerned, i.e. (under **SC**) we have that $c(v_i) \in X$. We first assign a set $\sigma(a_i) \subseteq S$ to each alternative a_i in $X = \{a_1, a_2, \ldots a_m\}$. It gives the set of members of S who properly choose a_i: $\sigma(a_i) = \{v_j \in S\text{: } c(v_j) = \{a_i\}\}$. Put $\Sigma = \{\sigma(a_1), \sigma(a_2), \ldots \sigma(a_m)\}$. Note that under **SC** the sets $\sigma(a_i)$ are mutually disjoint. At our fixed profile[1], let $k = \max(|\sigma(a_1)|, |\sigma(a_2)|, \ldots|\sigma(a_m)|)$: k represents the greatest number of votes received by an alternative. By the definition of π we have $\pi(S) = \{a_i\text{: } |\sigma(a_i)| = k\}$.

Since **SC** holds, if a voter does not properly choose some alternative, she must be unconcerned. Let $\Theta = \{v_j \in S\text{: } c(v_j) = X\}$. We can easily check that

$$S = (\bigcup\Sigma) \cup \Theta$$

where $\bigcup\Sigma = \{v_i\text{: there is some } a_r \text{ such that } v_i \in \sigma(a_r) \text{ and } \sigma(a_r) \in \Sigma\}$.

I shall make the following convention: write v^*_T for the voter $v_j \in T$ with the property that $j = \min(r\text{: } v_r \in T)$. Let $\Sigma^1 = \{\sigma(a_i)\text{: } \sigma(a_i) \neq \emptyset\}$.

Now we define a series of pick-up functions χ^j as follows[2]: $\chi^1(\sigma(a_i)) = v^*_{\sigma(a_i)}$ for each $\sigma(a_i) \in \Sigma^1$. So, for each alternative a_i with the property that it was properly voted by at least one voter, i.e. $\sigma(a_i) \neq \emptyset$, the function χ^1 selects one voter in $\sigma(a_i)$. Put also $\chi^1(\Sigma^1) = \{ v^*_{\sigma(a_i)}\text{: } \sigma(a_i) \in \Sigma^1\}$. We may observe that $\chi^1(\Sigma^1)$ has individual disjoint choices, and so by **DISC** and **SC** we get: $C(\chi^1(\Sigma^1)) = \{c(v^*_{\sigma(a_i)})\text{: } v^*_{\sigma(a_i)} \in \chi^1(\Sigma^1)\}$. Intuitively, $C(\chi^1(\Sigma^1))$ is the set of alternatives voted by at least one member of the group S.

Secondly, let $\sigma^1(a_i) = \sigma(a_i) - \{ v^*_{\sigma(a_i)}\}$ for each $i = 1, 2, \ldots m$. Define also $\Sigma^2 = \{\sigma^1(a_i)\text{: } \sigma^1(a_i) \neq \emptyset\}$. Let $\chi^2(\sigma^1(a_i)) = v^*_{\sigma^1(a_i)}$ for each $\sigma^1(a_i) \in \Sigma^2$. Following the same steps as above, we obtain $C(\chi^2(\Sigma^2)) = \{c(v^*_{\sigma^1(a_i)})\text{: } v^*_{\sigma^1(a_i)} \in \chi^2(\Sigma^2)\}$. Intuitively, $C(\chi^2(\Sigma^2))$ is the set of alternatives voted by at least two members of the group S. Clearly, $\chi^1(\Sigma^1) \cap \chi^2(\Sigma^2) = \emptyset$. Take also into account the fact that for all a_i, if there is some $v_j \in \chi^2(\Sigma^2)$ such that $c(v_j) = \{a_i\}$, then there is also some $v_{j'} \in \chi^1(\Sigma^1)$ such that $c(v_{j'}) = \{a_i\}$. Since both $\chi^1(\Sigma^1)$ and $\chi^2(\Sigma^2)$ have individual disjoint choices, **DISC** entails that $C(\chi^2(\Sigma^2)) \subseteq C(\chi^1(\Sigma^1))$. Applying **SAR** to $\chi^1(\Sigma^1) \cap \chi^2(\Sigma^2) = \emptyset$ and $C(\chi^2(\Sigma^2)) \subseteq C(\chi^1(\Sigma^1))$ we get $C(\chi^1(\Sigma^1) \cup \chi^2(\Sigma^2)) = C(\chi^2(\Sigma^2))$.

Further, we appeal to the same procedure to construct functions $\chi^3, \ldots \chi^k$ and also the other notions we used. Note that $\sigma^{k+1}(a_i)$ is empty for all i, and so we cannot define χ^{k+1}. By iteratively applying axioms **DISC** and **SAR**, we get:

$$C(\chi^1(\Sigma^1) \cup \chi^2(\Sigma^2) \cup \ldots \chi^k(\Sigma^k)) = C(\chi^k(\Sigma^k)).$$

Intuitively, $C(\chi^k(\Sigma^k))$ is the set of alternatives voted by k members of S and thus $C(\chi^k(\Sigma^k)) = \pi(S)$. Moreover, observe that $\chi^1(\Sigma^1) \cup \chi^2(\Sigma^2) \cup \ldots \chi^k(\Sigma^k) = \bigcup\Sigma$.

[1] Remember that n is the total number of voters in G; so a set $\sigma(a_i)$ may contain at most n members.

[2] Pick-up functions like χ are very important in set theory. They are called choice functions. However, in order not to confuse things, here I preferred to call them with a different name. When the domain of such functions is infinite, their existence is guaranteed by a special axiom of Zermelo–Fraenkel's set theory: the Axiom of Choice.

The final step of the proof is to take into account the set $\Theta = \{v_j \in S: c(v_j) = X\}$ (if it is not empty). By **U** we have $C(\Theta) = X$. Then **SAR** gives $C((\bigcup\Sigma) \cup \Theta) = C(S) = C(\chi^k(\Sigma^k)) = \pi(S)$. ■

Now let us introduce a new property of social choice functions. Although seldom explicitly formulated, it has a strong intuitive support:

Proper Choice (**PC**). If $a \in C(S) \neq X$, then there is some $v_i \in S$ such that $c(v_i) = \{a\}$.

PC states that an alternative cannot be included in the choice set of a concerned group if it was not properly chosen by at least one member of that group. The following proposition shows that π satisfies **PC**:

Proposition 2. If **SC** holds, then **SAR** and **DISC** entail **PC**.

Proof. Suppose that an alternative *a* is not properly voted by any $v_i \in S$. Then $\sigma(a) = \emptyset$. We only need to observe that the set $\sigma(a)$ is not a member of any Σ^i, and so *a* never appears in the choice set of any group constructed in the proof of Theorem 1a. The final step of the proof appealed to the set of unconcerned individuals. However, they either do not count in settling the social choice (if there is a concerned voter in *S*), and so *a* also does not count; or all the voters in *S* are unconcerned, and therefore by **U** the group *S* is also unconcerned. But in this case **PC** is trivially valid.

Finally we move to part (b) of Theorem 1. As noted above, since **BA** entails **SC**, its proof is immediate. However, let us try to see what happens with the properties used to characterize the majority choice function μ if **BA** holds.

First, consider **SAR**. If $X = \{a, b\}$, for each group *S*, its choice set $C(S)$ can be either a singleton or *X*. In the former case the group is concerned; in the latter one it is unconcerned. So, if both groups *S* and *T* that occur in **SAR** are concerned, then either their choice sets are identical and thus **SAR** is immediately true by **U**; or their chice sets are disjoint and so the **SAR** is vacuously true, because its premises are false. On the other hand, if both *S* and *T* are unconcerned, again **SAR** is true by **U**. But suppose that one group is concerned and the other is not. Then **SAR** assures us that the group resulting by merging them is concerned (and follows the choice of the concerned group). To put it differently, **SAR** states that unconcerned groups do not count. Its meaning reduces to that of the so-called property of Independence of an Unconcerned Coalition (Xu, Zhong: 2010):

Independence of an Unconcerned Coalition (**IUC**). If $S \cap T = \emptyset$ and $C(T) = X$, then $C(S \cup T) = C(S)$.

Second, consider axiom **DISC** and suppose that $S = \{v_i, v_j\}$, i.e. it includes only two individuals. Under **BA**, it entails the following property:

Simple Equal Treatment (**SET**). If $c(v_1) = \{a\}$, $c(v_2) = \{b\}$, then $C(\{v_1, v_2\}) = \{a, b\} = X$.

If the agenda consists in exactly two alternatives *a* and *b* (i.e. **BA** holds) and the two members of a group have opposite choices, then the group will be unconcerned.

Remember also that, by Proposition 1.2, **SAR** and **DISC** entail **U**. So under **BA** we also have that **IUC** and **DISC** entail **U**. Moreover, **IUC** and **SET** are entailed under **BA** by our axioms. Xu, Zhong (2010) proved that the following axioms characterize the simple majority rule: **F**, **SET**, **IUC** and **U**[3]. Here I showed that the axioms **SAR** and **DISC** entail (under **BA**) these four axioms.

5. Conclusion

The fact that single-profile axiomatizations of the plurality and majority rules can be obtained alongside the better known multi-profiles axiomatizations should not be surprising. As Pollak (1979) conjectured, ”it is likely that there are single profile analogues to virtually all the results in the theory of Social Choice”. Unfortunately, there are no algorithms to produce these counterparts. Therefore, as Sen (1977) noted, proofs that a multi-profile theorem has an inter-profile counterpart are not trivial. Rubinstein (1984) showed that the conjecture is true for a class of theorems which is characterized by a specific linguistic structure; he also showed by means of an example that the conjecture fails in some cases. The single-profile counterparts were usually produced by replacing references to multiple profiles with an appeal to a larger set of alternatives, or agenda.

In this paper I did not appeal to such extended agendas. The simple majority rule and the plurality rule were axiomatized in a unified framework by appealing to a set of intuitive properties that express intra-profile connections between individual and group choices, or connections between collective choices made by different groups. I also argued that in both cases some restrictions on the general frame (e.g. the number of available alternatives) or on the form of choice functions (**SC** and **BA**) play an important role.

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[3] Actually, in their paper **F** is called Self-Determination. They also appeal to a monotoniciy axiom, but show that it can be replaced by **U**.

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