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Characterizing majority rule: from profiles to societies

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Abstract

K. May characterized majority rule as a function satisfying anonymity, neutrality, and responsiveness. Recent work criticized his characterization and opened the way to the introduction of properties defined by taking into account an entire set of societies. Following this approach, this paper presents a new axiomatization of majority rule that appeals, besides a variant of May's responsiveness, to new properties I will call "null society" and "subsets decomposability".

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1. Introduction

In his influential *Econometrica* article, May (1952) characterized majority rule as a function satisfying three properties: anonymity, neutrality, and responsiveness. Following his work, many recent efforts aimed at defining different sets of necessary and sufficient properties of majority rule. The criticism was mostly focused on May's responsiveness axiom. Maskin (1995) replaced it with an axiom to the effect that if at some profile, majority rule does not generate a transitive social ordering, then no other social

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rule may generate it. Following this approach, Campbell and Kelly (2000), Aşan and Sanver (2002), and Woeginger (2003) presented axiomatizations that do not appeal to responsiveness.

The logical form of anonymity, neutrality and responsiveness is: given a society H , for each profile of it, there must be another profile the properties of which are related to the properties of the initial one. A society is defined as a set of individuals, each of them endowed with a preference relation. May's result consists then in proving that the majority rule defines certain transformations on the set of a society's profiles. The proposed recent axiomatizations appeal to some properties that exhibit quite a different logical form. Aşan and Sanver's weak path independence and Woeginger's reducibility to subsocieties state that, for each profile of a society H , there is a set of societies and some profiles of them, whose properties are related to the properties of the profile of the initial society. These axiomatizations do not appeal to responsiveness, and anonymity is also avoidable.

The main contribution of this paper is to present an axiomatization of majority rule that replaces profiles-definable properties with societies-definable ones. The paper will appeal to a variant of May's responsiveness axiom. While avoiding anonymity, neutrality is also preserved. However, it can easily be interpreted as connecting a society's profile with another society's profile in which the preferences are reversed. Most work on social welfare functions focus on profile-definable properties. My characterization of majority rule has an important intuitive flavor, and is intended to show that definability in terms of a set of societies is both simple and effective.

2. The framework

Let $N = \{j_1, \dots, j_n\}$ be a set of individuals; we assume that each individual j in N is endowed with a complete and transitive preference relation R_j on a set A of alternatives. For the purposes of this paper, it is sufficient to consider only two alternatives x and y . I will write $R_j = 1$ (resp. $R_j = -1$) whenever the voter j strictly prefers x to y (resp. y to x), and $R_j = 0$ when j is indifferent between the two alternatives; then $R_j \in \{-1, 0, 1\}$. A society is a subset H of N . Observe that this definition leaves open the possibility that a society has no members. The preferences of the members of a society H can be collected in a profile vector $R_H = (R_{k_1}, \dots, R_{k_h})$, with k_1, \dots, k_h the members of H , and $R_H \in \{-1, 0, 1\}^h$ defines the preference profile of the society H .

A social welfare function for the set N of individuals is a function $F: \cup_{k \leq n} \{-1, 0, 1\}^k \rightarrow \{-1, 0, 1\}$. Function F gives the aggregate preference for any preference profile of any society H . Now, let $H = \{k_1, \dots, k_h\}$ be a society and $R_H = (R_{k_1}, \dots, R_{k_h})$ a profile vector of it. I shall write $F(H, R_H)$ for $F(R_{k_1}, \dots, R_{k_h})$. However, whenever the reference to a certain profile vector is implicit, I shall write simply $F(H)$. If H_1, \dots, H_m is a collection of societies, then we can also define an aggregate preference $F(F(H_1), \dots, F(H_m))$ for this new 'society', the elements of which are exactly these sets: first, we aggregate the preferences of the members of each society, and get the preference (a complete and transitive ordering of the set A of alternatives) of that society; then we aggregate these preferences. We can thus define social functions not only for societies the members of which are individuals, but also for societies of societies¹.

¹ And, of course, for societies of societies of societies, etc. In general, we need not then sharply distinguish between individuals and societies, for the ur-elements of our framework may well be themselves societies.

Some properties of F can be stated with reference to one society and one profile of it. Pareto optimality (PO) is an example. Other properties are usually stated with reference to one society and two profiles of it. Neutrality (N) and May's positive responsiveness (PR) are examples. Still others can be stated by appealing to some new societies, different from the society function F is defined for. Woeginger's reducibility to subsocieties (RS) is an example. The characterization of the majority rule I will present in the next section appeals only to properties of the third type.

- (PO) For any society H and any profile vector $R_H=(R_{k_1}, \dots, R_{k_h})$ of it, if $R_k \geq 0$ (resp. $R_k \leq 0$) for all $k \in H$ and $R_k=1$ (resp. $R_k=-1$) for some $k \in H$, then $F(H)=1$ (resp. $F(H)=-1$).
- (N) For any society H and any profile vector $R_H=(R_{k_1}, \dots, R_{k_h})$ of it, there is some profile $R'_H=(-R_{k_1}, \dots, -R_{k_h})$ of it and $F(H, R'_H)=-F(H, R_H)$.
- (PR) Let H be a society and $R_H=(R_{k_1}, \dots, R_{k_h})$ a profile vector of it such that $F(H, R_H) \geq 0$ (resp. $F(H, R_H) \leq 0$). Then there exists a profile vector $R'_H=(R'_{k_1}, \dots, R'_{k_h})$ of H and an individual k_j in H such that for all $k_i \neq k_j$ $R'_{k_i}=R_{k_i}$ and $R'_{k_i} > R_{k_i}$ (resp. $R'_{k_i} < R_{k_i}$) and $F(H, R'_H)=1$ (resp. $F(H, R'_H)=-1$).
- (RS) For any society H and any profile vector $R_H=(R_{k_1}, \dots, R_{k_h})$ of it, there are h societies H_{-k_j} resulting by removing the individual k_j from H (and of R_{k_j} from R_H) and $F(H)=F(F(H_{-k_1}), \dots, F(H_{-k_h}))$.

3. The majority rule

I shall first introduce an assumption concerning the social welfare function in case the society we wish to consider has no elements. It seems natural that such a society \emptyset neither assents nor dissents on any issue. However, if we still want that to have F defined in this case, the only reasonable option is to put $F(\emptyset)=0$. One reason for introducing this assumption stems from a desire for completeness. We may want to define a social function for each society. This is useful when for some purposes, one takes into account Boolean operations on societies; intersection is in some cases empty, while we still may want to make sense of the operation. Another reason is that the null society assumption helps in proving some desirable properties. For example, I shall prove that it helps define an intuitive result on F in case H contains exactly one element (Lemma 1).

Null society assumption (NSA): If $H=\emptyset$, then $F(H)=0$.

The next three properties are societies-definable. They include neutrality, rephrased as a property of this type.

Subsets decomposability (SD). A social function F for a society $H=\{k_1, \dots, k_h\}$ ($h \geq 1$) is subsets decomposable iff: $F(H)=F(F(H_1), \dots, F(H_m))$, where H_j ($1 \leq j \leq m$) is a proper subset of H .

Additive responsiveness (AR). Let H be such that $F(H) \geq 0$ (resp. $F(H) \leq 0$), and let $j \notin H$. Then $F(H \cup \{j\})=1$ (resp. $F(H \cup \{j\})=-1$) if $R_j=1$ (resp. $R_j=-1$).

Neutrality (N). For any society $H=\{k_1, \dots, k_h\}$, there is some society $H'=\{k'_1, \dots, k'_h\}$ such that $R_{H'}=(R_{k_1}, \dots, R_{k_h})=(-R_{k_1}, \dots, -R_{k_h})$, and $F(H)=-F(H')$.²

² Observe that this formulation of N is stronger than the standard one: to see this, simply take $k_1=k'_1, \dots, k_h=k'_h$.

The core of this section consists in the Proof of Theorem 1, which offers a characterization of majority rule in terms of our three societies-definable properties. I have also added a proof of PO and the description of a procedure to derive function F in case society H is a singleton.

Lemma 1. *If a social welfare function F satisfies NSA, AR and N, then $F(\{j\})=R_j$.*

Proof. By NSA, $F(\emptyset)=0$, and hence, it satisfies the condition in the antecedent of AR. If $R_j=1$ (resp. $R_j=-1$), then $F(\emptyset \cup \{j\})=F(\{j\})=1$ (resp. $F(\{j\})=-1$) by AR. For $R_j=0$, neutrality yields the desired result.³ \square

Lemma 2. *If a social welfare function satisfies NSA, AR and N, then it satisfies PO.*

Proof. We can prove by induction on the cardinality of society H that if $R_k \geq 0$ for all $k \in H$ and $R_k = 1$ for some $k \in H$, then $F(H) = 1$. For $n = 1$, we immediately get the result by AR and NSA. Now assume that $n \geq 2$ and that we have already proved our proposition for $n - 1$. Let $|H| = n - 1$, $j \notin H$ and $R_j \geq 0$. Then $|H \cup \{j\}| = n$. First, observe that for any society $H' = k_1, \dots, k_n$ if for all h , $R_{kh} = 0$, then $F(H') = 0$ (this requires N). We have two cases:

- If $R_j = 1$ holds, then since $F(H) \geq 0$, we get $F(H \cup \{j\}) = 1$ by AR.
- If $R_j = 0$, then by supposition there is some j' in H such that $R_{j'} = 1$. Let $H' = (H \cup \{j\}) - \{j'\}$. Obviously, $|(H \cup \{j\}) - \{j'\}| = n - 1$, and $F((H \cup \{j\}) - \{j'\}) \geq 0$. By AR, $F(H \cup \{j\}) = F(((H \cup \{j\}) - \{j'\}) \cup \{j'\}) = 1$. \square

Theorem 1. *If NSA holds, then a social welfare function F satisfies AR, N and SD if and only if it is the majority rule (MJ).*

Proof. Necessity is straightforward. The only interesting case is to show that majority rule MJ satisfies SD, i.e. it is subsets decomposable⁴. Proof: suppose that the society H consists in k individuals, such that p of its members have $R_j = 1$, z of its members have $R_j = 0$, m of its members have $R_j = -1$, and $k = p + z + m$. What we want to show is that $\text{MJ}(H) = \text{MJ}(\text{MJ}(H_1), \dots, \text{MJ}(H_r))$, with $H_j (1 \leq j \leq r = 2^k - 1)$ the subsets of H . First note that $|H_j| \leq k$. Let $P_s(H)$ be the set of all sets H^* such that $|H^*| = s$ (of course, $s \leq k$). We obviously have: if $s = 0$, then $\text{MJ}(H^*) = 0$. Let us write H as: $\{i_1, \dots, i_p, j_1, \dots, j_z, k_1, \dots, k_m\}$. Suppose first that $p > m$. Then for each s there is no majority of sets H^* such that $|H^*| = s$ and $\text{MJ}(H^*) = -1$. Suppose that some H^* is such that $|H^*| = s$ and $\text{MJ}(H^*) = -1$ (if there is no such H^* , then $\text{MJ}(H^*) \geq 0$, and since $p > 0$ there is at least one H^* such that $\text{MJ}(H^*) = 1$); hence, the majority of subsets with cardinality s have the property that $\text{MJ}(H^*) = 1$. We can prove that there is a one-to-one correspondence between these sets and sets H'^* with cardinality s , but for which it holds that $\text{MJ}(H'^*) = 1$. Write H^* as: $\{i_1, \dots, i_{p_1}, j_1, \dots, j_{z_1}, k_1, \dots, k_{m_1}\}$, with $p_1 \leq p$, $z_1 \leq z$, $m_1 \leq m$. Then there is a subset H'^* of H such that $|H'^*| = s$ and $\text{MJ}(H'^*) = 1$. H'^* is defined by: $\{i_1, \dots, i_{m_1}, j_1, \dots, j_{z_1}, k_1, \dots, k_{p_1}\}$; it is easy to check that $\text{MJ}(H'^*) = 1$. Hence, there is no majority of sets with cardinality s and $\text{MAJ}(H^*) = -1$. But there is at least one s , namely 1, for which the majority of subsets of H with cardinality s is such that $\text{MJ}(H^*) = 1$. Hence, $\text{MJ}(\text{MJ}(H_1), \dots, \text{MJ}(H_k)) = 1$. The other two possible cases, when $p = m$ and $p < m$, do not raise other problems, and are left as an exercise for the reader.

³ Lemma 1 can be proved without NSA (see in this sense Woeginger, 2003): one only needs neutrality and PO, which, as Lemma 2 shows, is a consequence of our assumptions. However, the Proof of Lemma 2 requires NSA, and thus in our frame we cannot dispense with it.

⁴ By SD, one can equivalently ask if a majority of individuals in a society assent to some alternative, or if a majority of all possible subsocieties assent to that alternative.

To prove sufficiency, consider some social welfare function F that satisfies NSA, AR, N and SD. By induction on n , we have in this case $F = MJ$. For $n = 1$, Lemma 1 gives the result. Now suppose that $n \geq 2$ and that sufficiency was proved for all $n' < n$. We may distinguish three cases. First, let $p > m$. Then there is some individual j such that $R_j = 1$. Consider the society $H - \{j\}$. We have at it that $p \geq m$ and by induction $F(H - \{j\}) \geq 1$. By AR, $F((H - \{j\}) \cup \{j\}) = F(H) = 1 = MJ(H)$. Analogously, we can prove the case when $p < m$. Finally, let $p = m$. If $p = m = 0$, the theorem is proved, since we have $F(H) = F(R_1, \dots, R_m) = F(0, \dots, 0) = 0$. Suppose that $p = m > 0$. Again we have two cases, according as $z = 0$ or $z \geq 1$.

If $z \geq 1$, then there is some j such that $R_j = 0$. But $|H - \{j\}| = n - 1$, and by induction $F(H - \{j\}) = 0$. Then by AS, we get $F((H - \{j\}) \cup \{j\}) = 0 = F(H) = MJ(H)$. Now let $z = 0$ (i.e. nobody abstains). It is not difficult to show that for any subsociety H_1 ($|H_1| = k_1$) such that p_1 of its members have $R_j = 1$ and m_1 of its members have $R_j = -1$, and $k_1 = p_1 + m_1$, there is one and only one subsociety H_2 ($|H_2| = k_1$) such that m_1 of its members have $R_j = 1$ and p_1 of its members have $R_j = -1$, and $k_1 = p_1 + m_1$. By neutrality, $F(H_1) = -F(H_2)$. But then in the subsets decomposition of F , if $k_1 \geq 1$, then if $F(H^*) = v \in \{-1, 0, 1\}$ for some argument H^* , then there is exactly one argument H'^* of F such that $F(H'^*) = -v$ (for $k_1 = 0$, we have as an argument of F : $F(\emptyset) = 0$). Then, by neutrality $F(F(H_1), \dots, F(H_r)) = 0$ ($r = 2^k - 1$). By SD, $F(H) = 0 = MJ(H)$.⁵ \square

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⁵ This proof suggests that SD entails a variant of anonymity. Suppose we have a profile of H . If the preferences of the individuals are permuted, then the value of F at H for this new profile is unchanged. This conclusion yields immediately from SD, once we observe that the subsets decompositions in the two cases are identical.