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Actuality and World-Indexed Sentences

Abstract. Some logical properties of modal languages in which actuality is expressible are investigated. It is argued that, if a sentence like 'Actually, Quine is a distinguished philosopher' is understood as a special case of world-indexed sentences (the index being the actual world), then actuality can be expressed only under strong modal assumptions. Some rival rigid and indexical approaches to actuality are discussed.

Key words: modal logic, actuality indexed sentences.

In this paper I will investigate some logical properties of modal languages in which actuality is expressible. Usually, such languages result by extending a modal language with a new operator corresponding to the English adverb 'actually'. When applied to a sentence φ of the language, it produces a new one, actually φ. For example, consider the sentence:

$$\text{Quine is a distinguished philosopher.} \tag{1}$$

By applying to it the operator 'actually', we get another sentence:

$$\text{Actually, Quine is a distinguished philosopher.} \tag{2}$$

The question is, what is the logic of this operator? One way or another, the standard move is that of correlating this operator with the 'actual world': roughly, the main, and normal, function of the actuality operator is thought of to help define the evaluation of the sentence in its scope with respect to the actual world. A good approximation of this idea is to take a sentence like (2) to mean that:

$$\text{In the actual world, Quine is a distinguished philosopher.} \tag{3}$$

On this analogy, in its primary sense 'actuality' points to a possible world, the one which, among the other worlds, enjoys the property of being actual. However, (3) is just a special case of a world-indexed sentence like:

$$\text{In world } w\text{, Quine is a distinguished philosopher.} \tag{4}$$

In general, if φ is a sentence, then that φ is the case at world w, i.e.: at w, φ (or $w\varphi$ for short) is a w-indexed sentence. So, sentences like: 'In world w, Quine is a distinguished philosopher', or even 'In world w', that Quine is

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a distinguished philosopher is the case in world w', etc. are world-indexed. The claim I will try to substantiate in that an account of the logic of world-indexed sentences is highly significant for understanding the logical behavior of actual-world-indexed sentences like (3).

However, the investigation faces a serious problem. Consider, for instance, the case of actual-world-indexed sentences like (3). On the one hand, when using the operator 'in the actual world' we consider what is going on in some world, the actual one. The sentence (3) is true if the state of facts that Quine is a distinguished philosopher obtains in the actual world. On the other hand, the language which contains world-indexed sentences like (3) and (4) is modal. A Kripke-type semantics for that language postulates a collection of entities usually called, again, 'worlds'. Sentences (1) and (3) are then true or false at each of these worlds. Now, from among them we can select a world which is actual. The state of affairs that Quine is a distinguished philosopher may obtain in some of them, for instance in the one we take as actual, but fail to obtain in many other worlds. The problem is, how can we be sure that we speak of the same worlds in the two cases? How can we be sure that the world we take into consideration when using the operator 'in the actual world', and the world which, among the semantically postulated ones, is actual, are one and the same entity?

My view is that we have no guarantee that a systematic connection between the two collections of worlds exists from the outset. I hold, on the contrary, that it takes a lot of effort to define conditions to the effect that the worlds syntactically considered are among the worlds semantically postulated. Hence, it is only under such special conditions that a theory of actuality can be developed.

In the first three sections I develop a semantics for modal languages in which world-indexed sentences are allowed. In sections IV and V I discuss the possibility of constructing a one-to-one correspondence between the worlds syntactically considered and the worlds semantically postulated. In the final section I return to actual-world-indexed sentences as a special case of world-indexed ones and discuss some approaches to the logic of actuality.

I

The modal language $\mathscr{L}$ we will study[1] includes a set S of sentence letters, and the logical symbols $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\equiv$ and $\Box$. In addition, $\mathscr{L}$ includes a set W of world symbols w, w', w'', etc. For each world symbol $w \in W$, let

[1] In the first three sections I resume results presented in "Worlds Within Worlds", in *Nordic Journal of Philosophical Logic* 1 (1997), 2, 26–40.

$\langle w \rangle$ be the world-indexing operator: 'in world w'. The idea is that if we prefix a sentence φ by the operator $\langle w \rangle$, we get another sentence, that φ is the case at w. The sentences of $\mathscr{L}$ are the members of the smallest set containing: (i) every sentence letter; (ii) expressions $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \equiv \psi$, whenever it also contains φ and ψ; (iii) expressions $\neg\varphi$, $\Box\varphi$ and $\langle w \rangle\varphi$, whenever it also contains φ. The only new element this definition brings is in point (iii) that, intuitively, world-indexing a sentence yields a sentence too. If φ is a sentence, and $\langle w \rangle$ is a world-indexing operator, then that φ is the case at w, i.e. $\langle w \rangle\varphi$, is also a sentence.

The world-symbol w and the world-indexing operator $\langle w \rangle$ are very different things. However, to simplify the notation, I will use the following convention: I will write $w\varphi$ instead of $\langle w \rangle\varphi$, keeping always in mind that in the sequence $w\varphi$, w is just short for $\langle w \rangle$. Hopefully, the context will prevent this ambiguity in our notation to cause misunderstandings.

To start with, it seems natural to advance the following minimal requirements on the behavior of w-indexed sentences, for each world w. The idea is to try to mimic in our language $\mathscr{L}$ usual constraints from standard possible worlds semantics. For example, $\neg\varphi$ is the case at a world w if and only if φ is not the case at w; $\varphi \vee \psi$ is the case at w if and only if φ is the case at w or ψ is the case at w, etc.

1.1. $\vdash w\neg\varphi \equiv \neg w\varphi$, for every w,

1.2. $\vdash w(\varphi \wedge \psi) \equiv (w\varphi \wedge w\psi)$, for every w,

1.3. $\vdash w(\varphi \vee \psi) \equiv (w\varphi \vee w\psi)$, for every w,

1.4. $\vdash w(\varphi \rightarrow \psi) \equiv (w\varphi \rightarrow w\psi)$, for every w,

1.5. $\vdash w(\varphi \equiv \psi) \equiv (w\varphi \equiv w\psi)$, for every w.

We will also add to these the requirement that all provable sentences hold for each w:

1.6. If $\vdash \varphi$, then $\vdash w\varphi$, for every w.

For the beginning, let the underlying modal logic be the standard system K. The theorems of K are the tautologies, all expressions of the form:

1.7. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$,

and all expressions deducible from them by detachment and necessitation: the rule that if $\vdash \varphi$, then $\vdash \Box\varphi$. For reasons that will become apparent in what follows, I will call LK (local K) the logic characterized by the above-mentioned conditions.

The following is an immediate theorem of LK:

1.8. $\vdash w\varphi \vee w\neg\varphi$.

For every sentence φ, either φ is the case at w, or $\neg\varphi$ is the case at w. Indeed, applying (1.6) to the tautology $\varphi \vee \neg\varphi$, we get $w(\varphi \vee \neg\varphi)$, and from (1.3) we get directly (1.8). Now, if we repeat the same steps starting with (1.8), we obtain:

1.9. $\vdash w'w\varphi \vee w'w\neg\varphi$, for every φ.

Roughly, (1.9) expresses the fact that for each sentence φ, the world w' says that either φ is the case at w, or $\neg\varphi$ is the case at w, i.e. that world w' creates within itself an image of what is going on at w. It reflects or mirrors w. The world w' says how w is related to every sentence φ. Observe, further, that nothing guarantees that the way w is mirrored in w' is identical with the way w "really" is. For it is possible that for some φ, $w'w\varphi$, but $w\neg\varphi$: world w' claims that φ is the case at w, while in fact φ is not the case at w. But suppose that according to w', φ is the case at w if and only if really φ is the case at w, for every φ; or, to put it more formally: $w'w\varphi \equiv w\varphi$, for every φ. Then the world w' reflects or mirrors w adequately: w' says that something is going on at w if and only if that something really is going on at w.

Let us rephrase all these in a more rigorous way. The most important concept we will use is that of a w-localization of a set Σ of sentences of $\mathscr{L}$. If Σ is such a set, its w-localization, $\mathrm{LOC}_w(\Sigma)$ for short, is defined by:

1.10. $\mathrm{LOC}_w(\Sigma) := \{\varphi : w\varphi \in \Sigma\}$.

Thus, $\mathrm{LOC}_w(\Sigma)$ is the set of all those sentences of $\mathscr{L}$ which, at Σ, are the case at w. Given (1.9), our expectation is that if Σ is LK-consistent and maximal, then $\mathrm{LOC}_w(\Sigma)$ will also be LK-consistent and maximal. The proof that this is indeed so is fundamental for all results from below.

1.11. (The local maximality lemma) If Σ is a LK-consistent and maximal set of sentences of $\mathscr{L}$, then so are the w-localizations of Σ.

Proof. First, $\mathrm{LOC}_w(\Sigma)$ is LK-consistent. For if it were inconsistent, then for some ψ we would have both $\psi \in \mathrm{LOC}_w(\Sigma)$ and $\neg\psi \in \mathrm{LOC}_w(\Sigma)$. But, according to definition (1.10), we would have both $w\psi \in \Sigma$ and $w\neg\psi \in \Sigma$. By (1.1), we would also have $\neg w\psi \in \Sigma$, and this would contradict the supposition that Σ is LK-consistent. Second, $\mathrm{LOC}_w(\Sigma)$ is maximal. If it were not so, then for some ψ, neither $\psi \in \mathrm{LOC}_w(\Sigma)$ nor $\neg\psi \in \mathrm{LOC}_w(\Sigma)$ would obtain. However, since (1.8) is a LK-theorem, $w\psi \vee w\neg\psi \in \Sigma$ and hence either $w\psi \in \Sigma$ or $w\neg\psi \in \Sigma$. But then, by definition (1.10), either $\psi \in \mathrm{LOC}_w(\Sigma)$ or $\neg\psi \in \mathrm{LOC}_w(\Sigma)$, which contradicts the supposition on $\mathrm{LOC}_w(\Sigma)$. ∎

Think, e.g., of Σ as maximally describing the way facts "really" are. Then, for each w, the w-localization of Σ is the way w says (at Σ) that facts are. Note that, although it is possible that for some w, the w-localization $\mathrm{LOC}_w(\Sigma)$ of Σ be exactly Σ, at LK we cannot prove that this is always the case, i.e., that (at Σ) some world is such that it describes facts as they "really" are. Now, let the w'-localization of $\mathrm{LOC}_w(\Sigma)$ be exactly $\mathrm{LOC}_{w'}(\Sigma)$, for some world w':

1.12. $\mathrm{LOC}_{w'}(\mathrm{LOC}_w(\Sigma)) = \mathrm{LOC}_{w'}(\Sigma)$

Obviously, $\mathrm{LOC}_{w'}(\mathrm{LOC}_w(\Sigma))$ is (at Σ) the reflection of world w' by w. Indeed, according to definition (1.10), $\varphi \in \mathrm{LOC}_{w'}(\mathrm{LOC}_w(\Sigma))$ iff $w'\varphi \in \mathrm{LOC}_w(\Sigma)$ iff $ww'\varphi \in \Sigma$, for every φ; on the other hand, $\varphi \in \mathrm{LOC}_{w'}(\Sigma)$ iff $w'\varphi \in \Sigma$. Then, (1.12) is equivalent to: $ww'\varphi \equiv w'\varphi) \in \Sigma$, for all φ, i.e. (at Σ) world w adequately reflects w'.

II

In this section I will present a possible worlds semantics for LK. A model for $\mathscr{L}$ is a structure $\mathfrak{C} = \langle K, R, \mathsf{F}, \mho \rangle$, where K is a set of indices, R is a binary relation on K, F is a function from $W \times K$ to K, and $\mho$ is a function which assigns a truth-value to each sentence, relative to each element k of K. The definition of $\mho$ is the standard one, surely with a new case for sentences of the form $w\varphi$, with w in W:

2.1. DEFINITION OF $\mho$.

(i) if φ is a sentence letter, then $\mho(\varphi, k) = 1$ or $\mho(\varphi, k) = 0$;

(ii) if φ is $\neg\psi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, k) = 0$;

(iii) if φ is $\psi \vee \xi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, k) = 1$ or $\mho(\xi, k) = 1$;

(iv) if φ is $\psi \wedge \xi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, k) = \mho(\xi, k) = 1$;

(v) if φ is $\psi \to \xi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, k) = 0$ or $\mho(\xi, k)) = 1$;

(vi) if φ is $\psi \equiv \xi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, k) = \mho(\xi, k))$;

(vii) if φ is $\Box\psi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, k') = 1$ for all k' such that $R(k, k')$;

(viii) if φ is $w\psi$, then $\mho(\varphi, k) = 1$ iff $\mho(\psi, \mathsf{F}(w, k)) = 1$.

Models for $\mathscr{L}$ differ from usual models in possible worlds semantics in that they contain function F. The idea is to let the worlds we use in $\mathscr{L}$ mimic the indices in K: whenever at k it is true that φ is the case at w, then at the element k' of K corresponding by F to w (with respect to k) the sentence φ must be true. It follows that w mirrors or reflects in k the element $k' = \mathsf{F}(w,k)$ in K; k' is reflected in k as w. In general, $\mathsf{F}(w,k)$ varies with k: in different k's an element k' is not reflected as some fixed world w. Hence, in different k's the collection of sentences φ such that φ is the case at the world w do not necessarily coincide.

I will say that a sentence φ is true in a model $\mathfrak{C} = \langle K, R, \mathsf{F}, \mho\rangle$, and write $\mathfrak{C} \vDash \varphi$ for this, if $\mho(\varphi, k) = 1$ for all k in K; and I will say that a sentence φ is LK-valid, and write $\vDash_{\text{LK}} \varphi$ in this case, if it is true in all models. (Whenever there is no danger of confusion, I will omit subscripts.)

A most important thing to emphasize is that usually the indices in K are called 'worlds'; but of course they are different from the old worlds w, w', w'', etc. we appealed to. Although the objective of this paper is to closely connect "worlds" like w, w', w'', etc. and "worlds" like k, k', k'', etc., at the present stage of investigation we have to carefully distinguish them. I will call "WORLDS" the elements of K, while the members of W will still be referred to as "worlds".

Now we can state the main result of this section:

2.2. THEOREM. $\vdash_{\text{LK}} \varphi$ iff $\vDash_{\text{LK}} \varphi$.

PROOF. First, sufficiency comes as a straight consequence of the following results:

2.3.1. if φ is a tautology, then $\vDash \varphi$,

2.3.2. $\vDash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,

2.3.3. if $\vDash \varphi$, then $\vDash \Box\varphi$,

2.3.4. $\vDash \neg w\varphi \equiv w\neg\varphi$,

2.3.5. $\vDash w(\varphi \wedge \psi) \equiv w\varphi \wedge w\psi$,

2.3.6. $\vDash w(\varphi \vee \psi) \equiv w\varphi \vee w\psi$,

2.3.7. $\vDash w(\varphi \to \psi) \equiv w\varphi \to w\psi$,

2.3.8. $\vDash w(\varphi \equiv \psi) \equiv (w\varphi \equiv w\psi)$,

2.3.9. $\vDash w\varphi \vee w\neg\varphi$,

2.3.10. $\vDash \neg(w\varphi \wedge w\neg\varphi)$,

2.3.11. if $\vDash \varphi$, then $\vDash w\varphi$ for each φ.

For example, to prove (2.3.4), let $\mathfrak{C}$ be a model and let k be a WORLD. Then: $\mho(\neg w\varphi, k) = 1$ iff $\mho(w\varphi, k) = 0$, iff $\mho(\varphi, \mathsf{F}(w,k)) = 0$ iff $\mho(\neg\varphi, \mathsf{F}(w,k)) = 1$ iff $\mho(w\neg\varphi, k) = 1$. To prove (2.3.6), observe that $\mho(w(\varphi \vee \psi), k) = 1$ iff $\mho(\varphi\vee\psi, \mathsf{F}(w,k)) = 1$ iff $\mho(\varphi, \mathsf{F}(w,k)) = 1$ or $\mho(\psi, \mathsf{F}(w,k)) = 1$ iff $\mho(w\varphi, k) = 1$ or $\mho(w\psi, k) = 1$ iff $\mho(w\varphi \vee w\psi, k) = 1$. To prove (2.3.11), suppose that

$\mho(\varphi, k) = 1$ for all k, but $\mho(w\varphi, k') = 0$ for some k'. Next, $\mho(w\varphi, k') = 0$ iff $\mho(\varphi, \mathsf{F}(w, k')) = 0$. But, since F is a function, $\mathsf{F}(w, k')$ is a WORLD k'', of which we supposed that $\mho(\varphi, k'') = 1$.

To prove the necessity part of Theorem 2.2, let us suppose that some φ is not LK-provable (i.e., $\vdash_{\mathrm{LK}} \varphi$ does not hold). We will show that there is some model $\mathfrak{C}$ such that $\mathfrak{C} \models \varphi$ does not hold. This is the case if for some WORLD k, $\mho(\varphi, k) = 0$. Now, if φ is not LK-provable, then the set $\{\neg\varphi\}$ is LK-consistent. Hence it can be extended to a LK-maximal consistent set Σ. The model $\mathfrak{C}$ is defined as follows: first, K is the set of all LK-maximal consistent sets. Obviously, $\Sigma \in K$. Second, $R(\Sigma', \Sigma'')$ holds iff the set of all sentences φ such that $\Box\varphi$ is in Σ' is included in Σ''. Third, put $\mathsf{F}(w, \Sigma') = \mathrm{LOC}_w(\Sigma')$. By the local maximality lemma, $\mathrm{LOC}_w(\Sigma')$ is in K. Finally, let $\mho(\varphi, \Sigma') = 1$ iff $\varphi \in \Sigma'$ whenever φ is a sentence letter. The proof consists in showing that for every sentence φ, $\varphi \in \Sigma'$ iff $\mho(\varphi, \Sigma') = 1$. The only difficult cases are for $\varphi = w\psi$ and $\varphi = \Box\psi$:

(i) φ has the form $w\psi$. Then: $\mho(w\psi, \Sigma') = 1$ iff $\mho(\psi, \mathsf{F}(w, \Sigma') = 1$ iff $\psi \in \mathsf{F}(\psi, \Sigma')$ iff $w\psi \in \Sigma'$. First, $\psi \in \mathsf{F}(\psi, \Sigma')$ entails $w\psi \in \Sigma'$. Suppose that $w\psi$ does not belong to Σ'. According to the definition of $\mathsf{F}(w, \Sigma')$ and the local maximality lemma, ψ does not belong to $\mathsf{F}(w, \Sigma')$. The converse results by a simple application of the definition of F.

(ii) φ has the form $\Box\psi$. Then: $\mho(\Box\psi, \Sigma') = 1$ iff for each Σ'', if $R(\Sigma', \Sigma'')$, then $\mho(\psi, \Sigma'') = 1$; iff for each Σ'', if $R(\Sigma', \Sigma'')$, then $\psi \in \Sigma''$; iff $\Box\psi \in \Sigma'$. The difficult step is to show that if $R(\Sigma', \Sigma'')$ entails $\psi \in \Sigma''$, for each Σ'', then $\Box\psi \in \Sigma'$. We will show that if $\Box\psi$ is not in Σ', then there is some LK-maximal consistent set Σ'' such that $R(\Sigma', \Sigma'')$ and ψ is not in Σ''. The set $\Gamma = \{\xi : \Box\xi \in \Sigma'\} \cup \{\neg\psi\}$ is LK-consistent. It can then be extended to a LK-consistent maximal set Γ'. One can easily see that for all ξ such that $\Box\xi \in \Sigma'$, $\xi \in \Gamma'$, and hence $R(\Sigma', \Gamma')$. But, since Γ' contains $\neg\psi$ and is consistent, it is not the case that $\psi \in \Gamma'$-contradiction. To complete the proof of Theorem 2.2, it is sufficient to show that φ is not true in $\mathfrak{C}$. Indeed, since Σ is in K, and $\neg\varphi \in \Sigma$, we have $\mho(\neg\varphi, \Sigma) = 1$, hence $\mho(\varphi, \Sigma) = 0$. ■

The following observation is important to keep in mind. Let the WORLD k' be such that there is some w of which it holds that $\mathsf{F}(w, k) = k'$. Then $\mho(w\varphi, k) = \mho(\varphi, k')$, for every sentence φ. To put it in other words, φ is true at k' iff it is true at k that φ holds at w. But in this case the WORLD k provides a reflection of k' in it, and specifically it reflects the WORLD k' as the world w. As it looks from k, the world w is an exact copy of k'. The case is in fact more general, since k creates within itself an image of every WORLD k' which is F-connected to k *via* some world w. Here we find a first sense in which it is possible to say that the semantics developed in this section is "local": each element of K (a WORLD) simulate other WORLDS by means of worlds. However, this does not entail that WORLD k creates inside it a full, and exact, image of all the WORLDS in K.

The question is, how would it be possible to create inside a WORLD such an image? The next three sections of this paper focus on developing a strategy to provide such an answer. In the next one I will show how it is possible to simulate the alternativeness relation R by means of a surrogate relation, defined in terms of the function $\digamma$. In sections IV and V I will investigate the possibility of creating, in each WORLD, an adequate image of all the other WORLDS, as well as of the alternativeness relation between them. The burden of the argument will consist in proving completeness results by taking into account only those models in which a one-to-one correspondence between WORLDS and worlds holds. Hence, in those models, each WORLD will create, by means of its worlds, a full, and adequate, image of the entire model. As a result, it will then be possible to view our semantics as "local" in second, and stronger sense, that each WORLD (in a model) succeeds in providing all information all WORLDS in the model carry.

III

If we think of worlds as entities which mimic WORLDS, it is natural to consider the relation between them and modalities. According to the clause (2.1vii), a sentence φ is necessary at some WORLD k iff it is true at all WORLDS alternative to k. But now the founding idea of possible worlds semantics enters the picture. The question is, would it be possible to define conditions for a sentence's φ being necessary at k in terms of worlds, not of WORLDS? Specifically, the following condition immediately comes into one's mind:

3.1. $\mho(\Box\psi, k) = 1$ iff for all worlds w, $\mho(w\psi, k) = 1$.

The WORLD k renders ψ necessary iff at k, according to all worlds w, it is the case that ψ.

In this section we will define conditions to the effect that (3.1) holds. To begin with, observe that the claim that for all worlds w, $\mho(w\psi, k) = 1$ is equivalent to each of the following expressions:

3.2a. $\forall w\ \mho(\psi, \digamma(w, k)) = 1$,

3.2b. $\forall w \forall k'(\digamma(w, k) = k' \rightarrow \mho(\psi, k') = 1)$,

3.2c. $\forall k'(\exists w\ \digamma(w, k) = k' \rightarrow \mho(\psi, k') = 1)$.

Looking at the antecedent $\exists w\ \digamma(w, k) = k'$ of (3.2c), it is crucial to note that to assert $\exists w\ \digamma(w, k) = k'$ is to assert that some binary relation holds between k and k'. Call R this relation. Intuitively, k and k' (in this order) are related by R iff k' is (adequately) reflected in k as some world. Then (3.2c) can be rephrased as:

3.2d. $\forall k'(\mathsf{R}(k,k') \to \mho(\psi,k')=1)$.

By (3.2d), a sentence ψ is necessary at a WORLD k iff it is true in every WORLD which is (adequately) reflected in k as some world.

Now, if we write (2.1vii) as: $\forall k'(R(k,k') \to \mho(\psi,k')=1)$, the analogy between it and (3.2d) is striking. The only difference is that while the antecedent of (2.1vii) appeals to the alternativeness relation R, in (3.2d) we find, instead, the reflection relation R. But then our objective of finding conditions which render a sentence necessary at some WORLD iff it meets the condition expressed in the right side of (3.1) reduces to finding, under what conditions $R(k,k')$ holds iff $\mathsf{R}(k,k')$ holds. A precise answer is given by Theorem 3.4 below. First, let LT (local T) be the logic resulting by adding to LK:

3.3.1. All sentences of the form $\Box\varphi \to w\varphi$, for all φ.

3.3.2. The rule: if $\vdash_{\mathrm{LT}} w\varphi$ for each w, then $\vdash_{\mathrm{LT}} \Box\varphi$.

Condition (3.3.1) is a local counterpart of the standard T-principle: $\Box\varphi \to \varphi$. The latter states that if a sentence φ is necessary, then φ is true (in the reference WORLD); with (3.3.1), we have: if a sentence φ is necessary, then φ holds at every world w. Condition (3.3.2) is the local counterpart of the necessitation rule. According to it, if $w\varphi$ is LT-provable for each world w, then $\Box\varphi$ must be LT-provable. The intuition behind (3.3.1) and (3.3.2) is that worlds are similar to WORLDS, and by imposing the two conditions an attempt is made to render, with respect to worlds, standard conditions on WORLDS. Then:

3.4. THEOREM. A sentence φ is LT-deducible iff φ is true in all models $\mathfrak{C}$ in which relations R and R hold for exactly the same arguments.

PROOF. To prove the theorem, I will use the substitution method.[2] Starting with condition (3.3.1), we will show that it defines:

3.3.1′. $\forall w \forall k\, R(k,\mathsf{F}(w,k))$.

Indeed, (3.3.1) can be translated into the following second order-predicate logic expression:

3.3.1a. $\forall P \forall k \forall w (\forall k'(R(k,k') \to P(k')) \to P(\mathsf{F}(w,k)))$.

[2] See J. van Benthem, "Correspondence Theory", in D. Gabbay, F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol. II, D. Reidel, Dordrecht, 1984, pp. 167–247.

By substituting $R(k,*) = *$ for $P(*)$, we get:

3.3.1b. $\forall k \forall w(\forall k'(R(k,k') \rightarrow R(k,k')) \rightarrow R(k, \mathsf{F}(w,k)))$.

(3.3.1′) results immediately once we observe that the antecedent of (3.3.1b) is an instance of a tautology. Now, by usual calculations, (3.3.1′) is equivalent to

3.3.1c. $\forall w \forall k \forall k'(\mathsf{F}(w,k) = k' \rightarrow R(k,k'))$

and further

3.3.1d. $\forall k \forall k'(\exists w\ \mathsf{F}(w,k) = k' \rightarrow R(k,k'))$,

i.e. $\mathsf{R} \subseteq R$. We got thus half of Theorem 3.4. The other half comes with some transformations on the translation of (3.3.2). Indeed, we have:

3.3.2a. $\forall P(\forall w \forall k\ P(\mathsf{F}(w,k)) \rightarrow \forall k \forall k'(R(k,k') \rightarrow P(k')))$.

Let us substitute $\exists w'\ \mathsf{F}(w',k) = *$ for $P(*)$:

3.3.2b. $\forall w \forall k \exists w'\ \mathsf{F}(w',k) = \mathsf{F}(w,k) \rightarrow \forall k \forall k'(R(k,k') \rightarrow \exists w'\ \mathsf{F}(w',k) = k')$.

Since the antecedent of (3.3.2b) is always true, we get:

3.3.2c. $\forall k \forall k'(R(k,k') \rightarrow \exists w'\ \mathsf{F}(w',k) = k')$,

and this states that $R \subseteq \mathsf{R}$. ■

At LT it is then possible to get $\Box\varphi$ at a WORLD k iff $\mho(\varphi,k') = 1$ for all k' such that $\mathsf{R}(k,k')$. Now, this retains the standard way of defining the truth-value of necessary sentences at a world, except that R replaces the more usual alternativeness relation R. Further, observe that the definition of R is only in terms of F, and does not depend upon the component R of the model. This suggests the possibility of letting a model $\mathfrak{C}$ be simply a structure $\langle K, \mathsf{F}, \mho \rangle$ and use (3.2d) to modify Definition 2.1(vii) as:

2.1vii′. If φ is $\Box\psi$, then $\mho(\varphi,k) = 1$ iff $\mho(\varphi,k')$ for all k' such that $\mathsf{R}(k,k')$.

Keeping in mind expressions (3.1) and (3.2), this manoeuver entails that $\mho(\Box\varphi,k) = 1$ is definable as: for all w, $\mho(w\varphi,k) = 1$. Thus, at k a sentence φ is necessary iff for each w the sentence φ is the case at w. Notice that if a modal logic contains LT, its semantics might be correspondingly simplified to using only models like $\mathfrak{C} = \langle K, \mathsf{F}, \mho \rangle$.

I will end this section by noting an important theorem of LT:

3.5. $\vdash_{\mathrm{LT}} \Box\varphi \rightarrow \neg\Box\neg\varphi$.

The proof appeals to (3.3.1). From $\Box\varphi \to w\varphi$ and $\Box\neg\varphi \to w\neg\varphi$ we get: $(\Box\varphi \wedge \Box\neg\varphi) \to (w\varphi \wedge w\neg\varphi)$, hence $(\Box\varphi \wedge \Box\neg\varphi) \to (w\varphi \wedge \neg w\varphi)$, which gives: $\neg(\Box\varphi \wedge \Box\neg\varphi)$; and this is in turn equivalent to (3.5). Remark: the semantical condition corresponding to (3.5) is: $\forall k \exists k'\ \mathsf{R}(k, k')$, i.e. R is serial[3].

IV

Let LM be the modal logic obtained by adding to LT the axiom:

4.1. $\vdash \Box\varphi \to \Box\Box\varphi$.

Semantically, it is known that condition (4.1) requires that relation R be transitive.

4.2. LEMMA. Let $\mathfrak{C} = \langle K, \mathsf{F}, \mho\rangle$ be a LM-model. Then for each w and w'' there is some w' such that for all φ, $\mho(ww''\varphi \equiv w'\varphi, k) = 1$.

PROOF. We know that (4.1) expresses the fact that R is transitive: if $\exists w_1\ \mathsf{F}(w_1, k) = k'$ and $\exists w_2\ \mathsf{F}(w_2, k') = k''$, then $\exists w_3\ \mathsf{F}(w_3, k) = k''$. Suppose that for some w_1 and w_2 we have: $\mathsf{F}(w_1, k) = k'$ and $\mathsf{F}(w_2, k') = k''$. Then for all φ we have $\mho(w_1\varphi, k) = \mho(\varphi, k')$, and $\mho(w_2\varphi, k') = \mho(\varphi, k'')$. By the definition of $\mho$ for sentences of the form $w\psi$, we also have: $\mho(w_1w_2\varphi, k) = \mho(\varphi, k'')$. Since R is transitive, we get that there is some w_3 such that for all φ, $\mho(w_3\varphi, k) = \mho(\varphi, k'')$. Hence for all φ, $\mho(w_1w_2\varphi, k) = \mho(w_3\varphi, k)$; or, to put it differently, for each w_1 and w_2 there is some w_3 such that for all φ, $\mho(w_1w_2\varphi \equiv w_3\varphi, k) = 1$. ■

Let us for each world w define a set $[w]_k$ by: $w' \in [w]_k$ iff $\mho(w\varphi \equiv w'\varphi, k) = 1$ for each sentence φ. The intuitive idea is that at the WORLD k in $\mathfrak{C}$ the worlds w and w' are indistinguishable. Further, consider a function μ which picks from each set $[w]_k$ an element of it: $\mu([w]_k) \in [w]_k$. Now, starting from an element $k \in K$, we will build a new model $\mathfrak{C}_{k,\mu} = \langle K_k, \mathsf{F}_{k,\mu}, \mho_{k,\mu}\rangle$ of LM with the property that in it WORLDS and worlds are one-to-one correlated. I will say that $\mathfrak{C}_{k,\mu}$ is a mirror model of LM. The components of $\mathfrak{C}_{k,\mu}$ are defined as follows:

(i) $K_k = \{k_w : w \in W\}$;

(ii) $\mho_{k,\mu}(\varphi, k_w) = 1$ at $\mathfrak{C}_{k,\mu}$ iff $\mho(w\varphi, k) = 1$ at $\mathfrak{C}$.

The set of the WORLDS of $\mathfrak{C}_{k,\mu}$ is defined such that its elements are one-to-one correlated with the worlds in W. A simple application of the definition of $\mho$ and of the local maximality lemma implies that the set of sentences

[3] This conclusion is of course an immediate consequence of the fact that F is a function.

φ such that $\mho_{k,\mu}(\varphi, k_w) = 1$ is maximally consistent, and hence that $\mho_{k,\mu}$ satisfies conditions (i)–(vi) of Definition 2.1. To show that it also satisfies conditions (vii) and (viii), it is necessary to define the function $\mathsf{F}_{k,\mu}$. We will proceed as follows:

(iii) $\mathsf{F}_{k,\mu}(w'', k_w) = k_{w'}$ iff there is some w''' such that: 1) $\mho(ww''\varphi \equiv w'''\varphi, k) = 1$ holds at $\mathfrak{C}$ for all φ; and: 2) $\mu([w''']) = w'$.

That $\mathsf{F}_{k,\mu}$ is indeed a function follows from Lemma 4.2, which guarantees that w''', and hence $[w''']$ exists, and from the definition of μ, which renders the unicity of w'. Now, observe that $\mho_{k,\mu}(w''\varphi, k_w) = 1$ iff $\mho(ww''\varphi, k) = 1$. By Lemma 4.2 and the definition of μ, we have $\mho(w'\varphi, k) = 1$, and hence $\mho_{k,\mu}(\varphi, k_{w'}) = 1$, which proves condition (viii) of Definition 2.1. Condition (vii′) follows easily once we keep in mind that at $\mathfrak{C}$ function $\mho$ was already defined for sentences φ of the form $w\psi$. Finally, we can define as usual the relation $\mathsf{R}_{k,\mu}$ by: $\mathsf{R}_{k,\mu}(k_w, k_{w'})$ iff there is some w'' such that $\mathsf{F}_{k,\mu}(w'', k_w) = k_{w'}$.[4] This completes the proof that $\mathfrak{C}_{k,\mu}$ is a model of LM.

Observe, however, that it is possible that a sentence φ be true at a model $\mathfrak{C} = \langle K, \mathsf{F}, \mho\rangle$ only at WORLDS k such that there is not a WORLD k' so that $\mathsf{R}(k', k)$, and false otherwise. Call φ $\mathfrak{C}$-basic if this is the case. Then, obviously, there is no mirror model $\mathfrak{C}_{k,\mu}$ of LM such that φ is true at it. Also, call a sentence φ LM-basic if for every model $\mathfrak{C}$ of LM, φ is $\mathfrak{C}$-basic. Then:

4.3. THEOREM. Let φ is not LM-basic. Then $\{\varphi\}$ is LM-consistent iff there is some mirror model $\mathfrak{C}_{k,\mu} = \langle K_k, \mathsf{F}_{k,\mu}, \mho_{k,\mu}\rangle$ of LM such that $\mho_{k,\mu}(\varphi, k_w) = 1$ for some $k_w \in K_k$.

PROOF. If $\{\varphi\}$ is LM-consistent and φ is not LM-basic, then there is some LM-model $\mathfrak{C} = \langle K, \mathsf{F}, \mho\rangle$ such that $\mho(\varphi, k) = 1$ for some $k \in K$ of which it holds that $\mathsf{R}(k', k)$ for some k'. Then there is a world w such that $\mho(w\varphi, k') = 1$, and thus at $\mathfrak{C}_{k',\mu}$ we have $\mho_{k,\mu}(\varphi, k_w) = 1$. Next, by the completeness theorem for LM, if there is some model $\mathfrak{C} = \langle K, \mathsf{F}, \mho\rangle$ such that $\mho(\varphi, k) = 1$ for some $k \in K$, then $\{\varphi\}$ is LM-consistent. Suppose, conversely, that at $\mathfrak{C}_{k,\mu} = \langle K_k, \mathsf{F}_{k,\mu}, \mho_{k,\mu}\rangle$ we have $\mho_{k,\mu}(\varphi, k_w) = 1$ for some k_w. Then at $\mathfrak{C}$ it holds that $\mho(w\varphi, k) = 1$. But then, if $\mathsf{F}(w, k) = k'$, then $\mho(\varphi, k') = 1$, and $\mathsf{R}(k, k')$, i.e. $\{\varphi\}$ is LM-consistent and φ is not LM-basic. ∎

V

Unfortunately, Theorem 4.3 is not very rewarding. First, the result is stated with respect to a special class of sentences. But, second, at LM we cannot

[4] Exercise: show that $\mathsf{R}_{k,\mu}$ is transitive and serial.

prove that basic sentences really exist or not. There are of course various ways to deal with these points. In this section I will investigate the alternative of rejecting the possibility that a WORLD is such that it has no R-antecedent.[5] To do this, consider the following two conditions:

5.1. $\vdash \Box\varphi \rightarrow \varphi$.

5.2. If $\vdash \Box\varphi$ then $\vdash \varphi$.

Call LM1, respectively LM2, the two logics obtained by adding (5.1), respectively (5.2), to LM. The reason why we define logics LM1 and LM2 is that under conditions (5.1) and (5.2) at each model no WORLD k is such that there is no k' so that $\mathsf{R}(k', k)$ holds, and consequently no sentence is basic. To see this, we will start with (5.1). A standard result is that (5.1) defines the condition that relation R is reflexive. This entails that for each WORLD k there is some k', namely k itself, such that $\mathsf{R}(k', k)$. Condition (5.2) defines exactly the condition that for each k there is some k' such that $\mathsf{R}(k', k)$. This can be seen by substituting in the second-order translation of (5.2):

5.3.1. $\forall P((\forall k \forall k''(\mathsf{R}(k, k'') \rightarrow P(k'')) \rightarrow \forall k\, P(k))$

expression $P(*)$ by $\exists k'\, \mathsf{R}(k', *)$. We get:

5.3.1$'$. $(\forall k \forall k''(\mathsf{R}(k, k'') \rightarrow \exists k'\, \mathsf{R}(k', k'')) \rightarrow \forall k \exists k'\, \mathsf{R}(k', k)$.

Since the antecedent of (5.3.1$'$) is a logical truth, we immediately obtain:

5.3.1$''$. $\forall k \exists k'\, \mathsf{R}(k', k)$.

Hence at LM1 and LM2 if a sentence φ is consistent, then for no model $\mathfrak{C}$ it is the case that φ is $\mathfrak{C}$-basic. It is then easily provable a completeness theorem for LM1 which appeals only to mirror models:

5.4. THEOREM. The set $\{\varphi\}$ is LM1-consistent iff there is some mirror model $\mathfrak{C}_{k,\mu} = \langle K_k, \mathsf{F}_{k,\mu}, \mho_{k,\mu} \rangle$ of LM1 such that $\mho_{k,\mu}(\varphi, k_w) = 1$ for some $k_w \in K_k$.

As usual, let us say that φ is LM1-valid, and write $\models_{\text{LM1}} \varphi$ for this, if φ is true in all models of LM1. Further, let us say that φ is LM1-mirror valid

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[5] Another alternative would be to accept basic sentences, *via* e.g., provability logic. Although very tempting, I will not be concerned in this paper with it. (For provability logic and its connections with modal logic, see G. Boolos, *The Logic of Provability*, Cambridge University Press, Cambridge, 1993; C. Smorynski, *Self-Reference and Modal Logic*, Springer Verlag, New York, Berlin, 1985, etc.) The alternative is not very easy, because of the incompatibility between (3.5) and Löb's axiom:

(L) $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$.

Indeed, with condition (L) it is provable that $(\Box\varphi \rightarrow \neg\Box\neg\varphi) \equiv \neg\Box\bot$, and given (3.5), we would also have $\vdash \neg\Box\bot$. To work out this alternative would then require to relax the conditions on F, and specifically the condition that it be a function (see also footnote 3).

(and write $\models_{\text{LM1(M)}} \varphi$ for this) if φ is true in all mirror models of LM1. Now, a corollary of Theorem 5.4 is that:

5.5. $\vdash_{\text{LM1}} \varphi$ iff $\models_{\text{LM1}} \varphi$.

Our logic LM1 has the property we looked for: the sentences provable in it are exactly those sentences true in the models in which worlds and WORLDS are one-to-one correlated.

However, with respect to LM2 we cannot prove an analogous result. The reason is that the rule (5.2) does not hold at mirror models of LM2. For suppose that for all $k_w \in K_k$, $\mho_{k,\mu}(\Box\varphi, k_w) = 1$, but there is some $k_{w''}$ such that $\mho_{k,\mu}(\neg\varphi, k_{w''}) = 1$. Then for all w, $\mho(w\varphi, k) = 1$. Hence we have $\mho(\Box\varphi, k) = 1$, and by (4.1), $\mho(\Box\Box\varphi, k) = 1$. We will show that this is equivalent to: for all w and w', $\mho(ww'\varphi, k) = 1$. Indeed, we have: $\mho(\Box\Box\varphi, k) = 1$ iff for all w, $\mho(w\Box\varphi, k) = 1$; iff for all w and k', if $\mathsf{F}(w, k) = k'$, then $\mho(\Box\varphi, k') = 1$; iff for all w and k', if $\mathsf{F}(w, k) = k'$, then for all w', $\mho(w'\varphi, k') = 1$; iff for all w and w', $\mho(w'\varphi, \mathsf{F}(w, k)) = 1$; iff for all w and w', $\mho(ww'\varphi, k) = 1$. On the other hand, from $\mho_{k,\mu}(\neg\varphi, k_{w''}) = 1$ we get $\mho(w''\neg\varphi, k) = 1$. Thus, it holds that for all w and w', $\mho(ww'\varphi, k) = 1$, but $\mho(w''\neg\varphi, k) = 1$, for some w''. The argument is blocked, because we cannot use Lemma 4.2 to get a contradiction: we cannot be sure that w'' itself is among the worlds such that $\mho(w''\varphi, k) = 1$ iff $\mho(ww'\varphi, k) = 1$, for some pair w and w' of worlds.

We can, though, proceed as follows: let LM3 the logic obtained by adding to LM the conditions:

5.2.1. If $\vdash \Box\varphi$ then $\vdash \varphi$,

5.2.2. $\Box\Box\varphi \to \Box\varphi$.

It is not difficult to prove that (5.2.2) defines condition:

5.2.2′. $\forall k \forall w \exists w' \exists w'' \forall \varphi\ \mho(w\varphi \equiv ww'\varphi, k) = 1$.

Indeed, it is known that (5.2.2) defines the density of relation R: if $\mathsf{R}(k, k')$, then there is some k'' such that $\mathsf{R}(k, k'')$ and $\mathsf{R}(k'', k')$. Now, let w be a world and k a WORLD.[6] Then there is some k' such that $\mathsf{F}(w, k) = k'$. Obviously, this entails that $\mathsf{R}(k, k')$. Given the definition of $\mho$, we have for all φ, $\mho(w\varphi, k) = \mho(\varphi, k')$. Since R is dense, there are some k'', w' and w'' such that $\mathsf{F}(w', k) = k''$ and $\mathsf{F}(w'', k'') = k'$. Then for all φ, $\mho(w'\varphi, k) = \mho(\varphi, k'')$ and $\mho(w''\varphi, k'') = \mho(\varphi, k')$. Hence for all φ, $\mho(w'w''\varphi, k) = \mho(\varphi, k')$. From this and $\mho(w\varphi, k) = \mho(\varphi, k')$, we get: $\mho(w'w''\varphi, k) = \mho(w\varphi, k)$.

[6] If we accept (5.2.2) and in the same time no WORLD is reflexive, then the set W must of course be infinite, if we want to get a mirror model of LM3.

Now, using (5.2.2′) we immediately see that rule (5.2) holds at mirror models. Then:

5.6. THEOREM. The set $\{\varphi\}$ is LM3-consistent iff there is some mirror model $\mathfrak{C}_{k,\mu} = \langle K_k, \mathsf{F}_{k,\mu}, \mho_{k,\mu}\rangle$ of LM1 such that $\mho_{k,\mu}(\varphi, k_w) = 1$ for some $k_w \in K_k$.

An easy corollary is that:

5.7. $\vdash_{\text{LM3}} \varphi$ iff $\models_{\text{LM3}} \varphi$.

In the remainder of this paper I will focus on mirror models, while situations in which usual models of the logics considered are appealed to will be explicitly mentioned. Therefore, for simplicity I will omit subscripts attached to functions F and $\mho$. It is also extremely important to note that at these mirror models we can always correlate worlds and WORLDS: a WORLD k_w obviously corresponds to a world w; and, conversely, a world w obviously corresponds to a WORLD k_w. We can then simplify the notation and write, e.g., $\mathsf{F}(w, w') = w''$ instead of $\mathsf{F}_{k,\mu}(w, k_{w'}) = k_{w''}$, and $\mho(w\varphi, w') = 1$ instead of $\mho_{k,\mu}(w\varphi, k_{w'}) = 1$.

However, this notational convention leaves unanswered the question: How is a WORLD k_w related to a world w? To sketch an answer, let $\mathfrak{C}$ be some mirror model of LM1, and let w be one of its WORLDS. Then we can prove the theorem:

5.8.1. THEOREM. There is some w' such that for all φ, $\mho(w'\varphi, w) = 1$ iff $\mho(\varphi, w) = 1$.

PROOF. We already saw that the axiom (5.1) of LM1 defines the reflexivity of the relation R. But $\mathsf{R}(w, w)$ is equivalent to: for some w', $\mathsf{F}(w', w) = w$. Then for an arbitrary φ, we have: $\mho(\varphi, w) = 1$ iff $\mho(\varphi, \mathsf{F}(w', w)) = 1$, iff $\mho(w'\varphi, w) = 1$. ∎

Theorem 5.8.1 states that each WORLD w is reflected in itself as some world w'. But we can prove at LM1 something even stronger, that at w each worlds reflects some WORLD of the model:

5.8.2. THEOREM. For each world w'' there is some WORLD w' such that for each sentence φ, $\mho(\varphi, w') = 1$ iff $\mho(w''\varphi, w) = 1$.

PROOF. Keep in mind that we reason at mirror models. Theorem 5.8.2 holds iff at the original model we have: for all w and w'' there is some w' such that for every φ, $\mho(w'\varphi, k) = 1$ iff $\mho(ww''\varphi, k) = 1$, i.e. $\mho(w'\varphi \equiv ww''\varphi, k) = 1$, which is obviously true under Lemma 4.2. ∎

But note, first, that by Theorem 5.8.1 at w we have no reasons to suppose that the world which is a reflection of WORLD w at itself is exactly w. Second, although by Theorem 5.8.2 at w each world w'' is a reflection of some WORLD w', we cannot hold, conversely, that each WORLD is reflected in w as some world. Third, we have no guarantee that WORLD w reflects WORLD w' as w' really is. In the remainder of this section I will focus on the first of these questions.[7]

By Theorem 5.8.1 we are naturally tempted to take the world which reflects a WORLD w at itself be exactly w. To do this, we have to move from: there is some w' such that $\mathsf{F}(w', w) = w$ to:

5.9. $\mathsf{F}(w, w) = w$,

i.e. WORLD w self-mirrors[8]. However, what is the syntactical counterpart of condition (5.9)? To detect it, I will use a back-and-forth argument. Roughly, it runs as follows: start with a semantical condition at a mirror model. Then move to the original model, and see if it defines there a syntactical condition. Since the mirror model, the original model and the generating WORLD were arbitrarily chosen, conclude that the newly found condition holds for every model, and hence, by completeness, that it is a theorem. Finally, move forth, by the completeness theorem for mirror models, to the truth of that condition in each mirror model. In our case, if (5.9) holds, then for each sentence φ we must have $\mho(w\varphi \equiv \varphi, w) = 1$. At the original model and WORLD, we have: $\mho(ww\varphi \equiv w\varphi, k) = 1$. So we will put

5.10. $\vdash ww\varphi \equiv w\varphi$

which must then be true at every mirror model. I will call LM4 the logic resulting by adding axiom (5.10) to LM1.

VI

In the previous sections I gave an account of how to answer the main problem of a theory of world-indexed sentences: *How is it possible to correlate worlds syntactically considered and* WORLDS *semantically postulated?* On this ba-

[7] The answer to the other two questions is still open. For example, it is possbile to argue as follows: the condition that each WORLD w' is reflected as some world w'' is equivalent with: $\forall w \forall w' \exists w''\ \mathsf{F}(w'', w) = w'$, and further with: $\forall w \forall w'\ \mathsf{R}(w, w')$. Now, to assure that it holds, it seems that, because of (4.1) and (5.1), we only need to add to LM4 the axiom:

5.11. $\varphi \rightarrow \Box\Diamond\varphi$

and get in this way a new logic LM5. Obviously, its underlying modal logic is the standard S5. However, I am not sure that, when attempting to prove completeness, we face no problem due to the interference between this condition and F's being a function.

[8] In "Worlds Within Worlds", p. 28, I called w self-conscious.

sis, I will come back and concentrate on the favorite case of world-indexed sentences: those prefixed by the operator 'in the actual world'. To be sound, our investigation should provide an account of how "actual world" denotes a certain WORLD, the one that is *actual* (whatever this might mean). I will present two approaches to this issue. The former is syntactically motivated. The idea is to select from the members of the set W of our language $\mathscr{L}$ a special operator α, taken to mean: 'in the actual world', and study it in analogy with the other world-indexing operators. So, suppose that we are at LM1 and that α is one of the elements of W. Then, by appealing to the axioms of this logic, we have, e.g.:

6.1.1. $\alpha\neg\varphi \equiv \neg\alpha\varphi$,

6.1.2. $\alpha(\varphi \wedge \psi) \equiv \alpha\varphi \wedge \alpha\psi$,

6.1.3. if $\vdash \varphi$, then $\vdash \alpha\varphi$,

6.1.4. $\Box\varphi \rightarrow \alpha\varphi$,

etc. Are there other properties of the α operator? Well, it depends upon the view on actuality one accepts. Consider, indeed, the following two propositions:

6.2. $w\alpha\varphi \equiv \alpha\varphi$,

6.3. $w\alpha\varphi \equiv w\varphi$.

With the former proposition, world-indexing is parasitic upon α-indexing: it brings nothing new about the status of a α-indexed sentence. Contrariwise, with the latter proposition α-indexing is superfluous with respect to other indexing operators: if φ is to be indexed, then it does not matter if it was already α-indexed. The two propositions, as we will immediately see, diverge in a very deep way: each is consistent with one of two competing views on understanding actuality, a rigid and, respectively, an indexical view.[9]

THE RIGID VIEW. According to it, αreflects at each world one and the same world, the actual one. And it does so rigidly, that is, whatever world of evaluation we choose, "α" always sends us to one and the same world α. Formally, we have:

6.4. $F(\alpha, w) = \alpha$, for every w.

[9] D. Lewis defends an indexical view in "Anselm and Actuality", in *Philosophical Papers*, vol. I, Oxford University Press, Oxford, 1983; A. Plantinga, in *The Nature of Necessity*, Clarendon Press, Oxford, 1974, argues for a rigid view. See also G. Forbes, *Languages of Possibility: An Essay in Philosophical Logic*, Blackwell, Oxford, 1989, for a useful account of the current debate on the issue.

To see what syntactical counterpart (6.4) has, I will appeal again to a back-and-forth argument. We get, for every φ, $\mho(\varphi, \alpha) = 1$ iff $\mho(\varphi, \mathsf{F}(\alpha, w)) = 1$, iff $\mho(\alpha\varphi, w) = 1$. Moving back to the initial WORLD and model, we get $\mho(\alpha\varphi \equiv w\alpha\varphi, k) = 1$, from which it results that (6.2) must be a theorem, and hence that it is true at the mirror model.

Thus, on the rigid account φ is actually true at a world w iff it is actually true, i.e. true at the actual world. The operator αalways invokes the *actual world.* Changing the world of evaluation rigidly keeps unchanged the world "α" points to.[10] The intuitive idea behind (6.2) is that α is a backward-looking operator: the current world w at which we want to observe what actually is the case is parasitic upon the actual one.

The actual world has a very interesting property: the operator α is redundant when the actual world is the world of evaluation:

6.5. $\alpha\varphi \equiv \varphi$

is true at α, i.e. $\mho(\alpha\varphi \equiv \varphi, \alpha) = 1$. Quine is a distinguished philosopher iff actually Quine is a distinguished philosopher, is bound to hold at the actual world.[11] However, (6.5) need not hold at other worlds: for it is only actually (= at the actual world) true, not necessarily.

THE INDEXICAL VIEW. On this view, α will reflect at each world that very world. At w, α reflects w itself; at w', α reflects w', etc. This way, what world is actual depends upon the evaluation point: being actual is nothing but being the evaluation point. Once we change the evaluating point, what world plays the role of the actual world will also change. To get this result, it suffices to put:

6.6. $\mathsf{F}(\alpha, w) = w$, for each w.

Again, a back-and-forth argument shows that (6.6) makes proposition (6.3) true at each mirror model. Since (6.3) is in turn equivalent to:

6.3.1. $w(\alpha\varphi \equiv \varphi)$

it follows that propositions:

6.3.2. $\Box(\alpha\varphi \equiv \varphi)$

6.3.3. $\alpha\varphi \equiv \varphi$

[10] However, note that at this moment we have no guarantee that $\alpha$is unique.

[11] This can be expressed formally by:

3.5.2. $\Box\alpha(\alpha\varphi \equiv \varphi)$,

which is a theorem under (6.2).

are true at all worlds in all models. The difference from the rigid view is then striking: on that view, $\alpha\varphi \equiv \varphi$ holds only at the actual world; on the indexical one, it holds at every world.

Both views agree, however, that the logic of actuality is best captured by specifying from the set W of worlds of our language $\mathscr{L}$ a world-indexing operator α, the behavior of which is modeled following that of the other world-indexing operators in W. But it is, though, possible to pursue a different, semantical, approach to actuality. According to it, we need not specify from the collection of world-indexing operators 'in the world w', some operator to mean 'in the actual world'. Rather, we may try to define semantical conditions to the effect that some world will play the role of the actual one. To see how this approach works, remember that at LM1 axiom (5.1): $\Box\varphi \rightarrow \varphi$ defines the standard property of the reflexivity of the alternativeness relation R: for every w, $\mathsf{R}(w, w)$, i.e.:

6.7. $\exists w'\ \mathsf{F}(w', w) = w$, for every w.

Thus, for each world w, there is some adequate reflection w' of it in itself: we have $\mho(w'\varphi \equiv \varphi, w) = 1$ for every sentence φ. It is then difficult to resist taking w' be just the actual world at w. At w, the operator 'in the world w'' plays the role of: 'in the actual world'.[12] So, we will require that:

6.8. $\mathsf{F}(w, w) = w$, for every w

holds. This condition is of course stronger than (6.7), and it entails that for each sentence φ, $\mho(w\varphi \equiv \varphi, w) = 1$.[13]

It may look that this semantical approach enforces the indexical view on actuality, as opposed to the rigid one. For, indeed, in each world w, it is w which is the actual world. In different worlds, different worlds are actual. What 'in the actual world' points to depends upon the context of evaluation. This description is correct, though partial. For it overlooks one distinctive feature of the present semantical frame. Specifically, at logics including LM1 Theorem 5.9.1 holds at the original model $\mathfrak{C}$ and WORLD k too. Then it is true that for some world w, $\mho(\varphi, k) = 1$ iff $\mho(w\varphi, k) = 1$, for every sentence φ. But then at the mirror model generated by $\mathfrak{C}$ and k (and the function μ), there is some WORLD k_w such that $\mho(\varphi, k) = 1$ at the original model iff $\mho(\varphi, k_w) = 1$ at the mirror model. Now, by the definition of the mirror model, at the initial model the WORLD k creates within itself a full, and

[12] However, nothing prevents that for some φ, both $\mho(w'\varphi, w) = 1$ and $\mho(\varphi, w') = 0$ hold. Only at w, world w' looks like w; at some world w'', it might look quite different from w.

[13] Now, (6.8) is our old friend (5.9), and this defines expression: $ww\varphi \equiv w\varphi$. Once we adopt it, we move from LM1 to LM4.

adequate, copy of the entire mirror model. So, let w be the WORLD which reproduces at the mirror model the WORLD k. Then at w we have complete claims of what is going on at every world of the mirror model. Suppose, further, that at w the world w itself is the actual world. But then, since w is an exact copy of the entire model, it follows that w is in one sense the actual world not only at w, but also at the entire model. To assert that the present semantical approach favors an indexical or a rigid view on actuality is then ambiguous. If we consider the model as such, an indexical view looks to be the favorite. But if we look inside the WORLD which is the mirror reflection of the original WORLD, and keep in mind the fact that a full reflection of the entire model can be met therein, then a rigid view on actuality appears to be preferred.

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