ANALELE
UNIVERSITĂȚII
BUCUREȘTI

# FILOSOFIE

EXTRAS

ANUL XLII / 1993

# ON THE 'DERIVABILITY' CONDITION IN REDUCTION *

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Balzer, Moulines and Sneed offer in a recent project [1] an extensive discussion of the most important global intertheoretical relations. Among them, reduction occupies a central place of interest. It seems to me that the authors' approach to reduction is important for at least two reasons : firstly, a precise and workable concept of reduction is advanced ; and secondly, the treatment they offer of it provides the most explicit statement yet of the views the three leading champions of structuralism held at the end of the eighties on the role of language in the philosophical understanding of science. In this paper I try to scrutinize in more detail this later aspect. I start from the comments made by Balzer, Moulines and Sneed on the role of the 'derivability' condition in reduction, and come to the conclusion that some of them are at best ambiguous and that the attempt at formally reconstructing that condition in *An Architectonic for Science* is not a satisfactory one. Finally, a new reconstruction of the role of the 'derivability' condition in reduction relations is sketched.

## I

Let $T$ and $T^*$ be idealized theory-elements. For theory T to directly reduce to $T^*$ $(T \rho T^*)$, it is necessary that the following condition obtains :

(A) For all $x, x^*$: if $x^* \in M^*$ and $\langle x^*, x\rangle \in \rho$, then $x \in M$.

Here $M^*$ (respectively $M$) is the set of models of $T^*$ (respectively of $T$), and $\rho$ is a reduction relation, relating the potential models of the two theories ($\rho \subseteq Mp^* \times Mp$)[2]. As the authors put it, (A) expresses a general condition of derivability of the laws of $T$ from the laws of $T^*$, through the mediation of $\rho$[3]. Thus ,in informal terms, (A) is simply taken to mean that[4] :

(B) The laws of $T$ can be derived (under translation) from those of $T^*$.

* I would like to thank Professor W. Balzer for helpful comments and criticism on an earlier draft of this paper.

[1] W. Balzer, C. U. Moulines, J. D. Sneed, *An Architectonic for Science*, D. Reidel, Dordrecht, 1987.

[2] *Ibidem*, p. 277. (A) is in fact condition (3) of the definition DVI-5.

[3] *Ibidem*, p. 275.

[4] See also p. 308.

However, when trying to put in more precise terms the logical relations between (A) and (B), one meets many and sometimes unexpected problems. One is this. The general concept of reduction was defined so that it would not amount to, or even yield, a mutual translation of the languages of both theories. So, (B) can be understood in a way which does not presupose any reference to the languages of the theories $T$ and $T^*$. To do that, we have first to define a law for theory $T$. Say that a law for $T$ is a class $X$ of its potential models so that $M \subseteq X$; and analogously for $T^*$. Then, we have to take care that the 'derivability' of law $X$ from some law $X^*$ is mediated by $\rho$. The content of (B) can now be rendered by :

(C) For any law $X$ of $T$ there is some law $X^*$ of $T^*$ such that $X^* \| \models X$. Now our task comes to marking precise the meaning of the 'derivability' relation $\| \models$. Clearly, $\| \models$ cannot be defined by, say, $X^* \subseteq X$, for no connection is assumed between sets $Mp^*$ and $Mp$. There are, though, two intuitive requirements that have to obtain for $\| \models$ to hold :

(c. 1) $X^*$ must somehow depend upon $X$ and the reduction relation $\rho$.

(c. 2) Whenever a model $x^*$ of $T^*$ belongs to $X^*$, the corresponding model $x$ (via relation $\rho$) of $T$ belongs to $X$.

Let us try to analyse in more detail these requirements. First, observe that (c. 2) can be simply rendered by :

(c. 2') for all $x^*$ and $x$: if $x^* \in M^*$ and $x^* \in X^*$, then $\rho(x^*) = x$ is such that $x \in M$ and $x \in X$.
Second, since $X^*$ is that subset of $M^*$ containing all and only those $x^*$ which are $\rho$-related to at least some $x \in X$, (c. 1) comes to :

(c. 1') $X^* = \{x^* :$ there is some $x$ such that $x \in X$ and $\rho(x^*, x)\}$
We can define now the derivability relation $\| \models$ as follows : $X^* \| \models X$ iff $X^*$ and $X$ are so related that requirements (c. 1') and (c. 2') obtain. Consequently, (C) becomes :

(C') If $X$ is a law of $T$, then for all $x^*$, $x$: if $x^* \in M^*$ and there is some $x'$ such that $x' \in X$ and $\rho(x^*, x')$, then, if $\rho(x^*) = x$, $x \in M$ and $x \in X$.

Furthermore if we take $X$ to be the class $Mp$ of all the potential models of $T$, then, given that the quantifiers 'for all $x$' and 'there is some $x$' range over the members of $Mp$, we can omit any expression like $x \in X (= Mp)$. Thus, (C') simplifies to :

(C'') For all $x^*$, $x$: if $x^* \in M^*$ and there is some $x'$ such that $\rho(x^*, x')$, then, if $\rho(x^*) = x$, then $x \in M$.
It is important to note that the clause : 'there is some $x'$ such that $\rho(x^*, x')$' — or, a bit more formally, '$(\exists x') \rho(x^*, x')$' — states that $x^*$ is a member of the domain of the (partial) function $\rho$, i.e. $x^* \in \mathrm{Dom}(\rho)$. It is now apparent that, if the content of (B) is rendered model-theoretically (though as a special case) by (C''), then actually we have adopted the reading provided by Balzer, Moulines and Sneed [5] of (A) :

[5] *Ibidem*, p. 310.

$(A')\ (\forall x)\ (\forall x^*)\ (x^* \in M^* \cap \mathrm{Dom}(\rho) \rightarrow \rho(x^*) \in M)$

((C″) was slightly modified to let the quantifier 'for all $x$' be, contrary to its occurence in (A′), nonempty ; for in (A′) $x$ does not occur in the scope of $\forall x$).

To conclude, I do not agree with Balzer, Moulines and Sneed that (A′), and hence (C″), does not express a 'derivability' condition of the laws of $T$ from those of $T^*$. It does express one, but without any reference to languages. In this case, the phrase 'under translation' in (B) must be taken to refer to the connection between $M^*$'s and $M$'s *via* the reduction relation $\rho$. Note also that in fact (C″) and (A′) are logically equivalent to (A). So, if they are taken to express the meaning of (B), then the 'derivability' condition is simply *equivalent* to (B).

## II

However, Balzer, Moulines and Sneed seem to take the phrase 'under translation', and hence the whole expression (B), as involving linguistic considerations. They take 'translation' to refer to a relation between sentences of the language $L$ of $T$ and sentences of the language $L^*$ of $T^*$, and they conceive of laws as linguistic entities, which must be related by syntactic connections. Of course, if no committment to language is made, (A′) and (C*) might be regarded as equivalent to (B), i.e. to the claim that the laws of $T$ can be derived (under certain qualifications) from those of $T^*$. But, if we consider that $T$ and $T^*$ involve linguistic constructs, then, as Balzer, Moulines and Sneed put it, the content of (B) becomes [6] :

(D) $(\forall \alpha \in a)\ (trans^{-1}(a^*) \vdash \alpha)$

where $a$ and $a^*$ are the sets of axioms of $T$, and respectively of $T^*$, and *trans* is a function mappinp the sentences of $L$ to the sentences of $L^*$. Since there is no possibility of confusion here, I shall simply write $\vdash$ for $\vdash_L$. Our problem concerning the 'derivability' condition now comes to asking if, and under which conditions, (A′) (or, (C″)) is logically equivalent to (D).

Let me consider here the proof, summarized in TVI-15, that the two expressions are, under certain conditions, logically equivalent [7]. The proof is that (A′) is equivalent to (D) if the following conditions obtain :

$T$, $T^*$ are idealized theory elements with languages $L$, $L^*$ and $a$, $a^*$, *trans* and $\rho$ are such that :

1. a) $a \subseteq \mathrm{Sent}(L)$, $a^* \subseteq \mathrm{Sent}(L^*)$

   b) *trans* : $\mathrm{Sent}(L) \rightarrow \mathrm{Sent}(L^*)$

   c) $L$ and $L^*$ are first-order languages

   d) $\rho : Mp'' \rightarrow Mp$ is a partial function

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[6] This is implicit in the statement of theorem TVI-15, p. 310.
[7] *Ibidem*, pp. 309—311.

2. a) $\mathrm{Mod}(a) = M$, $\mathrm{Mod}(a^*) = M^*$
   b) $\mathrm{Rge}(\rho) = Mp$
   c) for all $\alpha \in \mathrm{Sent}(D)$ and all $x^* \in \mathrm{Dom}(\rho)$, $\rho(x^*) \models \alpha$ iff $x^* \models trans(\alpha)$
3. a) for all $\alpha^* \in a^*$ there is some $\alpha \in a$ such that $trans(\alpha) = \alpha^*$
   b) $trans^{-1}(a^*)$ is finite

I have stated the conditions (1) — (3) in a different way from the one given by Balzer, Moulines and Sneed in order to make clearer their import. Conditions (1) are structural, and show the logical nature of the concepts involved ; conditions (2) are more material, and involve the logical properties those concepts must have in order to be used in the formal reconstruction of, e.g., the relation of reduction. Conditions (3) are more unusual, being those special assumptions necessary if the logical equivalence of (A′) and (D) is to be proved. Thus, there should be no surprise if in what follows, I concentrate just on the use of (3a) and (3b) in the proof of TVI-15.

Condition (3a) is essentially involved in the proof of both the necessary and the sufficient parts of TVI-15. But I think that (3a) is highly unrealistic, as Balzer, Moulines and Sneed seem to admit themselves [8]. I can't immagine any genuine (i.e. non-symmetrical) reduction relation which would let all the counterparts of the axioms of $T^*$ be among the axioms of T. However, my main argument against (3a) is that it is not even necessary in the attempt to show that (B) conveys the same information as (A). I shall argue below that (D) is not a proper formulation of the language-theoretic content of (B), but let me first say a few words about (3b). This is a too strong condition, and in fact the proof of TVI-15 does not require it. Balzer, Moulines and Sneed use condition (3b) to show that (A′) implies (D) in the following way. Let $\alpha \in a$. It was already proved that :

$$\mathrm{Mod}(trans^{-1}(a^*)) \subseteq \mathrm{Mod}(\{\alpha\}).$$

Condition (3b) together with the completeness theorem for the first-order language $L$ seem to entail that $trans^{-1}(a^* \vdash \alpha$.But in this step of the proof the appeal to (3b) is not necessary. Indeed, the (generalized) completeness theorem for first-order logic is this [9] : if $\Sigma$ is a (finite or infinite) set of sentences, and $\alpha$ is a sentence, then $\mathrm{Mod}(\Sigma) \subseteq \mathrm{Mod}(\alpha)$ iff $\Sigma \vdash \alpha$. So, $trans^{-1}(a^*)$ need not be finite to yield desired result. I suspect that some confusion is involved in the proof given by Balzer, Moulines and Sneed of Theorem TVI-15. Recall that $trans^{-1}(\alpha^*)$, for $\alpha^* \in a^*$, is a sentence of $L$, and $trans^{-1}(a^*)$ is a class of sentences of $L$. If it is finite, then $\wedge trans^{-1}(a^*)$ is the conjunction of the sentences in $trans^{-1}(a^*)$, and it is indeed a sentence of $L$. Then, by completeness, $\mathrm{Mod}(\Sigma) \subseteq \mathrm{Mod}(\alpha)$ iff $\vdash \Sigma \rightarrow \alpha$, and in our case we get

[8] *Ibidem*, p. 308.

[9] See, e.g., C. C. Chang, H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973,

$D')\ (\forall\alpha\in a)\ (\vdash \Lambda trans^{-1}(a^*) \rightarrow \alpha)$ .

It is easy to prove now that (D′) implies (D) (the deduction theorem is essentally used here) ; but (D) implied (D′) (again by use of the deduction theorem) only if (3b) holds, i.e. if $trans^{-1}(a^*)$ is finite.

## III

I think that the account of the significance of TVI-15 is dramatically faulted by a misunderstanding of the meaning of (B). When it is stated in (B) that the laws of $T$ are to be derivable *under* translation from those of $T^*$, what is meant is not, of course, that we have to translate the laws of $T^*$ into laws of $T$; indeed it is misleading to concentrate upon the derivability of the laws of $T$ from those of $T^*$ in $T$! Rather, what (B) implies is that the laws of $T^*$ allow of the derivability of the translations (in $L^*$!) of the laws of $T$. To see that, let us first consider a syntactical analogue of (C). Syntactically, $T$ and $T^*$ are classes (deductively closed) of sentences. A law of $T$ is a sentence $X$ of $L$ such that $T \vdash X$; and analogously for $T^*$. We have also to take care that the 'derivability' of a law $X$ from some law $X^*$ is mediated by the reduction relation $\rho$. If theories $T$ and $T^*$ are constructed as linguistic entities, then the content of (B) can be rendered by :

(E) For any law $X$ of T there is some law $X^*$ of $T^*$ such that $X^* \Vdash X$.

Apart from the fact that in (E) the term 'law' refers to things quite different from those it referred to in (C), the only difference between (E) and (C) is that the 'derivability' relation '$\Vdash\!=$' has been replaced by the 'derivability' relation '$\Vdash$' ; the former is a semantical relation, the later must be conceived of as a syntactical one. How can we make '$\Vdash$' precise? Clearly, it cannot be defined directly in terms of $\vdash_{L^*}$, for $X^*$ is a sentence of $L^*$, while $X$ is a sentence of $L$; and, for the same rasons, it cannot be defined directly in terms of $\vdash_L$. But we can state, as we did in the case of $\Vdash\!=$, two intuitive requirements that have to obtain for $\Vdash$ to hold :

(e. 1) $X^*$ must somehow depend upon $X$ and the reduction relation $\rho$.

(e. 2) Whenever $X$ is derivable from $T$, $X^*$ must be derivable from $T^*$ too.

Now we can define the 'derivability' relation $\Vdash$: $X^* \Vdash X$ iff $X^*$ and $X$ are related so that conditions (e. 1) and (e. 2) — or appropriately reconstructed versions of these — obtain.

Observe, first, that the requirement (e. 2) states just that :

(e 2′) If $T \vdash_L X$, then $T^* \vdash_{L^*} X^*$

Since $X$ is arbitrary, we can infer that

(e. 2″) For all $X$, if $T \vdash_L X$, then $T^* \vdash_{L^*} X^*$

Recall that $T$ and $T^*$ are deductively closed classes of sentences, hence if (e. 2′) holds for any $\alpha \in a$, then it holds for all $X$ and consequently (e. 2″) holds also. So, suppose that

$(e_1)$ For all $\alpha \in a$: if $T \vdash_L \alpha$, then $T^* \vdash_{L^*} \alpha^*$

But clearly for any $\alpha \in a$, $T \vdash_L \alpha$ holds, and hence ($e_1$) is logically equivalent to :

($e_2$) For all $\alpha \in a : T^* \vdash_{L^*} \alpha^*$

Since, moreover, $T^*$ is deductively closed and is axiomatized by $a^*$, we have

($e_3$) For all $b^* : T^* \vdash_{L^*} b^*$ iff $a^* \vdash_{L^*} b^*$

By ($e_3$), ($e_2$) is logically equivalent to :

($e_4$) For all $\alpha \in a : a^* \vdash_{L^*} \alpha^*$

Surely, ($e_4$) entails (e. 2″).

Now, let us move to condition (e. 1). The first candidate for $X^*$ that comes to mind is simply *trans(X)*. Hence, (e. 1) could tentatively be made precise by :

(e. 1′) $X^* = trans(X)$

and in particular we would have $\alpha^* = trans(\alpha)$. If we take this route. we can now return to expression (E). We get a precise version of it, once we substitute $\alpha^*$ in ($e_4$) for $trans(\alpha)$ :

(E′) $(\forall \alpha \in a)\ (a^* \vdash_{L^*} trans(\alpha))$

(E′) can be nicely compared with (D). The difference between them is that while with (D) Balzer, Moulines and Sneed assumed that the 'translation' mentioned in (B) is a translation in $L$ of $T^*$, and hence the 'derivability' condition concers derivability *in* $T$, (E′) moves the whole enterprise inside $L^*$, and hence the deductive powers of $T^*$, not of $T$, are considered. I think that this move is in a much larger agreement with our intuitions about the reduction relations.

Balzer, Moulines and Sneed's approach could also be objected to on the following ground : if $X^*$ is constructed as *trans(X)*, then the only connection between theories $T$ and $T^*$ is provided by the function *trans*. But clearly this cannot be correct. In their informal comments on (B), Balzer, Moulines Sneed explicitly claimed that the derivation of the laws of $T$ from of $T^*$, under translations, must be conceived as being done *through the mediation of* $\rho$ ; and we accepted and worked with this mediation when we reconstruced in model-theoretical terms the 'derivability' condition as (C). However, nothing in (D) involves this mediation ; and the same happens with (E′) ! So, I suspect that the failure to syntactically reconstruct (B) as somehow involving the relation $\rho$ is one of the main reasons why the need arose for Balzer, Moulines and Sneed to posit such unrealistic and even implausible conditions as (3a) and (3b) to guarantee the equivalence of (A) to (reconstructed) (B).

It seems then that (e. 1) does not receive a proper understanding if it is reconstructed as (e. 1′). Indeed, if the law $X^*$ of $T^*$ corresponding to the law $X$ of $T$ is exactly the translation *(via* the function *trans)* of $X$, then there is no place for the reduction relation $\rho$ and for its mediating role. On the contrary, if $\rho$ is not to be neglected, (e. 1′) should be modified as :

(e. 1″) $X^* = S^* \rightarrow trans(X)$

where $S^*$ is some sentence corresponding in $L^*$ to the semantical relation $\rho$. I shall have more to say about $S^*$ in the next section. It should be noticed that, if (e. 1″) is preferred to (e. 1′), then a proper reconstruction of (E) would be, rather than (E′), the expression :

(E″) $(\forall \alpha \in a)\ (a^* \vdash_{L^*} S^* \rightarrow trans(\alpha))$

By the deduction theorem, we get :

(E‴) $(\forall \alpha \in a)\ (a^*, S^* \vdash_{L^*} trans(\alpha))$

and (E‴) is logically equivalent with (E″).

If (E‴), rather that (D) is taken as a proper reconstruction of the 'derivability' condition (B) — and languages in which the theories are formulated are considered — then the very problem to which TVI-15 tried to give an answer changes in the following sense : the task is to show that (A) amounts to 'derivability' (syntactically viewed), i.e. that (A) is logically equivalent to (E″). But since (A) was shown to be logically equivalent to a (semantical) 'derivability' condition (A′), the task will consist in a proof that (perhaps under certain conditions) (A′) simply means (E″).

The result I want to prove is summarized in the following

*Theorem* : If $T$ and $T^*$ are idealized theory-elements with languages $L$, $L^*$, and $a$, $a^*$, *trans* and $\rho$ are such that conditions (1a)—(1d), (2a)—(2c) and

(3c) There is some sentence $S^* \in L^*$ such that $\mathrm{Mod}(S^*) = \{x^* : (\exists x)\ \rho(x^*, x)\}$

hold, then

(A′) $(\forall x)\ (\forall x^*)\ (x^* \in M^* \cap \mathrm{Dom}(\rho) \rightarrow \rho(x^*) \in M)$

iff

(E″) $(\forall \alpha \in a)\ (a^* \vdash_{L^*} S^* \rightarrow trans(\alpha))$

Proof.

I. (E″) implies (A′)

(i) by the deduction theorem, (E″) yields

$(\forall \alpha \in a)\ (a^*, S^* \vdash_{L^*} trans(\alpha))$

Recall that (i) is in fact the expression (E‴).

(ii) by the completeness theorem for the first-order logic of $L$,

$\mathrm{Mod}(a^*) \cap \mathrm{Mod}(S^*) \subseteq \mathrm{Mod}(trans(\alpha))$, for all $\alpha \in a$.

(iii) $\{x^* : (\exists x)\ \rho((x^*, x)\} = \mathrm{Dom}(\rho)$

(iv) $\mathrm{Mod}(a^*) \cap \mathrm{Dom}(\rho) \subseteq \mathrm{Mod}(trans(\alpha))$, for all $\alpha \in a$ (from (ii) and (iii), by (3c))

(v) Suppose that $x^* \in \mathrm{Mod}(a^*) \cap \mathrm{Dom}(\rho)$

(vi) $x^* \in \mathrm{Mod}(\mathit{trans}(\alpha))$, for all $\alpha \in a$ (from (v), by (iv))

(vii) $x^* \models \mathit{trans}(\alpha)$, for all $\alpha \in a$ (from (vi))

(viii) since $x^* \in \mathrm{Dom}(\rho)$, there is some $x \in Mp$ such that $\rho(x^*) = x$

(ix) $\rho(x^*) \models \alpha$, for all $\alpha \in a$ (by (vii), (viii) and (2c))

(x) $\rho(x^*) \in M$ (from (ix))

II. (A′) implies (E″)

(xi) let $x^* \in M^* \cap \mathrm{Dom}(\rho)$. Since $x^* \in \mathrm{Dom}(\rho)$, $\rho(x^*)$ exists and it belongs to $Mp$.

(xii) $\rho(x^*) \in M$ (by (A′))

(xiii) let $\alpha \in a$. Since $\rho(x^*) \in M$, we get $\rho(x^*) \models \alpha$

(xiv) $x^* \models \mathit{trans}(\alpha)$ (by (xiii) and (2c))

(xv) $x^* \in \mathrm{Mod}(\mathit{trans}(\alpha))$

(xvi) if $x^* \in M^* \cap \mathrm{Dom}(\rho)$, trans $x^* \in \mathrm{Mod}(\mathit{trans}(\alpha))$ (from (xi) and (xv))

(xvii) since $x^*$ was arbitrary, we have shown that

$M^* \cap \mathrm{Dom}(\rho) \subseteq \mathrm{Mod}(\mathit{trans}(\alpha))$

(xviii) by the completeness theorem and (3c),

$a^*, S^* \vdash_{L^*} \mathit{trans}(\alpha)$

(xix) since $\alpha \in a$ was arbitrary, we get

$(\forall \alpha \in a)\ (a^*, S^* \vdash_{L^*} \mathit{trans}(\alpha))$

and, by the deduction theorem, (xix) — which in fact is (E‴) — implies (E″)

## IV

The proof of my substitute of TVI-15 heavily relies upon the use of the additional condition (3c). It seems necessary, therefore, to reflect a moment on its use in the two parts of the proof.

First, we have to observe that, formally speaking, there is no *a priori* objection to Dom(α)'s being represented as a sentence of $L^*$. For, indeed, the entities of which it makes sense to ask *if* they are semantical counterparts of the sentences of $L^*$ are collections of elements of $Mp^*$. And, though $\rho$ is a collection of pairs of potential models like $(x^*, x)$ with $x^* \in Mp^*$ and $x \in Mp$, $\mathrm{Dom}(\rho) = \{x^* : (\exists x)\ \rho(x^*, x)\}$ is a proper subset of $Mp^*$ and therefore it makes sense to ask if there is a sentence $S^*$ of $L^*$ such that Dom(ρ) is exactly the class of models of $S^*$. Intuitively, the models collected in a set like Dom(ρ) satisfy some property, and thus allow us to separate them. It is natural, I think, to take that property to consist in those conditions which the models of the reducing theory must satisfy in order to correspond in an appropriate sense to the models of the reduced theory. A paradigmatic instance of intertheory reduction in the relation between Rigid Body Mechanics (RBM) and Classical Particle Mechanics (CPM). From a structural perspective, a potential model of RBM consists of a domain comprising a single rigid body, together with functions which include mass, position, force, moment of inertia, etc. A potential model of CPM consists of a set of particles, and the basic concepts of CPM are represented as functions like mass, position and forces acting on the elements in the domain. An appropriate reduction relation between models of the two theories pairs the rigid body of the first model and the set of particles belonging to the second. The kinematics of the rigid body corresponds to the motion of the particles in the domain, and the various forces operating on the rigid body are conceived as corresponding to an appropriate composition of the components of the forces acting on the particles *

By (3c) there must exist a syntactical counterpart S* to this structural conditions. Roughly, S* states that all particles are rigid, i.e. that the relative distances of particles remain fixed over time. So, S* is a sentence

$$(\forall p)\ (\forall p')\ (\forall t)\ (\forall t')\ (/s(p, t) - s(p', t)/ = /s(p,t') - s(p', t')/)$$

of the reducing theory CPM.

Howerer, things are not as simple as they might appear to be. For the move to a syntactical fact — $S^*$ is a sentence of $L^*$ — is not entirely obvious, and, moreover, sometimes it might even be implausible. Consider the two steps in which condition (3c) was used in the proof of the theorem presented in the previous section. In (iv), condition (3c) was used to get, starting from a sentence $S^*$, a class Dom(ρ) of models, i.e. the move was from languages to (classes of) structures. Now turn to (xviii), where (3c) was used to produce a sentences $S^*$ corresponding to a class Dom(ρ) of structures. This move was from (classes of) structures to language.

The every sentence of a first-order language one can always find a class o structures of that language which contains all and only those structures in which that sentence is true. When applied to our special case, this basic fact of model theory comese to :

(F) if $\alpha^*$ is a sentence of the language $L^*$, then there is a class Mod($\alpha^*$) of all and only those $Mp^*$'s in which $\alpha$ is true.

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* This composition procedure can be defined, in the case of position and mass, by

R(m) = df. $\Sigma$ m(x) for all particles p.

R(s, t) = df. s(p, t) for some particle p such that the following holds :

for all t, $R(s, t) = \dfrac{\Sigma\, m(p) \cdot s(p, t)}{R(m)}$

The composition R(f) of forces acting on the particles must be constructed so that Newton's second law holds, i.e.

$R(f) =$ df. $R(m) \cdot D^2R(s, t)$.

Since (F) must be true, the proof of (iv) by (3c) is sound. It is important to note that in this case (3c) was used not to guarantee that a class of potential models of $T^*$, corresponding to $S^*$, must exist — for this job is done by (F) — but to indicate which is that class (= Dom($\rho$)). Hence, condition (3c) was used only as a means to inform us about the *meaning* of $S$. A more interesting use — and I suspect that this is the *proper* function of introducing (3c) — occurs with step (xviii). A well known result in model theory states that not all classes of structures of a certain language can be correlated with sentences of that language. Say that such a class is elementary if it can be correlated with a sentence, i.e. if it is the class of all the models of that sentence.

But there is no *a priori* reason to take the class Dom($\rho$) itself to be an elementary one, i.e. one that has a linguistic counterpart in $L$. However, here condition (3c) comes in. It assures us that Dom($\rho$) has such a counterpart. (3c) does not entail that whatever arbitrary classes of potential models of $T^*$ there may be are elementary ; it is concerned only with the class Dom($\rho$) and assures that *it* is elementary.

One might resist this line of argument by objecting that it is not clear why, for each pair of idealized theory-elements $T$ and $T^*$ and each reduction relation $\rho \subseteq Mp^* \times Mp$ we would like to consider, the class Dom($\rho$) should be elementary. For many pairs of theories and for many relations $\rho$ connecting their potential models, one might indeed find out that condition (3c) holds. The case of the reduction of RBM to CPM, as a matter of fact, fits condition (3c). But there are arbitrarily many ways to construct the reduction relation $\rho$, and we have no guarantee that Dom($\rho$) is always elementary. And if $S^*$ does not exist, no proof of the step (xviii) is offerred.

When faced with this objection, we could of course take the heroic route and maintain that (3c) is true : the sentences $S^*$ always exists. However, there are other, weaker, reactions to the objection. A first strategy is to try to turn all the occurrences of $S^*$ into empty ones. To do that, replace (3c) by :

(3c′) $M^* \subseteq \mathrm{Dom}(\rho)$

Theen the task of the theorem is to prove that (A′) is logically equivalent to :

(E′) $(\forall \alpha \in a)\ (a^* \vdash_{L^*} trans(\alpha))$

and in (E′) no reference to $S^*$ is considered. To prove the theorem, the only step in need of reelaboration is (xviii). But observe that (3c′) entails that

$$M^* \subseteq M^* \cap \mathrm{Dom}(\rho)$$

and this, together with

(xvii) $M^* \cap \mathrm{Dom}(\rho) \subseteq \mathrm{Mod}(trans(\alpha))$, for all $\alpha \in a$ entails

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which by the completeness theorem yields immediately (E′).

If $S^*$ were allowed to exist, then (3c′) would be equivalent (by completeness) to :

$a^* \vdash_{L^*} S^*$

But clearly this is not plausible, at least if the relation which the theory $T$ bears to $T^*$ is that of being *directly reducible.* (Indeed, if we take $T^*$ as a specialization, for which the reduction conditions expressed by $S^*$ hold, of some other theory $T^{*\prime}$, in this case we meet a case of *indirect* reduction of $T$ to $T^{*\prime}$).

Another reaction to the objection against $S^*$ is to accept that in the general case there is no linguistic counterpart of Dom(ρ), taken as such ; but emphasize that we could always handle, with linguistic means, all *contexts* in which Dom(ρ) occurs. For exemple, look again at (xviii). To provide a linguistic counterpart to it, I assumed that we were in need of sentences corresponding to $M^*$, to Dom(ρ) and to Mod(*trans*(α)).. And, while we already had the required sentences corresponding to the first and the third class of potential models of $T^*$ involved in (xviii), the only additional supposition we had to make seemed to be this : some sentence of $L^*$ should be correlated with Dom(ρ).

I think that this argument is not compelling. Indeed, if just one more sentence is needed, why not ask that it be correlated with Mod($a^*$)⋂Dom(ρ), rather than with Dom(ρ) ? If the intuitive idea that Dom(ρ) expresses some reduction conditions is retained, while the suggestion that Dom(ρ) determines a sentence is rejected, then it is still possible to regard Mod($a^*$)⋂Dom(ρ) as a restriction on the class of the structures which satisfy the axioms $a^*$.

To obtain such a restriction, an appealing idea is, of course, to call for a sentence $S^*$ which, when added to $a^*$, brings about the desired results. It is possible, though, to get the same results by taking a quite different route. Let $\alpha^*$ in $L^*$ be some logical consequence of $a^*$. For each such expression $\alpha^*$ one can get a collection Mod($\alpha^*$) of all its models. The idea of the present strategy is to claim that, if Mod($\alpha^*$) is elementary, then the collection Mod($\alpha^*$) ⋂ Dom(ρ) of models of $L^*$ is elementary too, i.e. there is a sentence $\alpha^*\rho$ the models of which are exactly the elements of Mod($\alpha^*$) ⋂ Dom(ρ). To get an intuitive grasp of what a sentence $\alpha^*(\rho)$ of this kind looks like, let me consider again the relation between CPM and RBM. Let, e.g., $\alpha^*$ be a sentence $(\forall p)\,\varphi(p)$ of CPM. Roughly speaking, $\alpha^*(\rho)$ is the assertion that $\varphi$ holds if restricted to rigid particles. To put it in more formal terms, $\alpha^*(\rho)$ is

$$(\forall p)(\forall p')((\forall t)(\forall t')(/s(p,t) - s(p',t)/ = /s(p,t') - s(p',t')/ \rightarrow \varphi(p'))$$

There is a very deep difference between the strategy based on (3c), and this strategy of restricting quantifiers occuring in the sentences of the reducing theory. Indeed, while in the first case it is supposed that one can get some sentence S*, which could then be used in reduction, in the second case no such sentence is provided. Rather, one has a systematic means to produce, starting from sentences of the reducing theory, those sentence of the same theory which help to devise the reduction relation.