MODAL AXIOMATIZATIONS OF THEORIES, RAMSEYFICATION AND THEORETICITY

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The aim of this paper is to develop an argument against the seman-al definitions of the theoretical. It is specifically directed against Bal-r's view, as expressed in (1); but I believe that it covers all attempts theoreticity which essentially involve the standard notion of a model a theory. It is argued that modal (or : modal and possibilistic) axioma-zations of theories could be considered with a view to dissolving this ›rt of argument. The most important result is that a *simple logical* ·*iterion* of the "theoretical in a given theory" works with modally (and pos-ibilistically) axiomatized theories.

I. Balzer's approach. Roughly speaking, his view is this : let B be , theory. Semantically, B is given as a class of models. A model of a theory s thought of as a typified structure in which there occur sets of objects ınd relations (in particular, functions) over those sets. A function f of B ıs B-definable iff the interpretation of f is uniquelly determined for all models of B. Now, f is called theoretical in a theory T if it can be defined in a subtheory B of T. Balzer notices that in some cases B-definability can be appropriately given by use of, e.g., equivalence "up to transformations of scale" or "up to liniar transformations". Second, he claims that, to be precise, the above definition of the theoretical needs a condition that T's invariances be respected. Let x and y be in B and let them differ at most in their f-components (write $x_f = y_f$ in this case); then $f_x = f_y$ (or : $f_x = \alpha f_y$) with, e.g., f_x the interpretation of f at x.

It should then be noted that the only invariants Balzer seems to take into account concern *relationships* which : 1) hold *among* different models of T; and 2) concern the *values* of the functions of T.

Commenting on the second of these requirements, Balzer notes (1, p. 135 n), that his definition of the theoretical has an easy and adequate interpretation in terms of "theory-guided" measurement (of the values of function f for some admissible argument). Balzer suggests that under such an interpretation his criterion can be nicely compared with Sneed's "informal" one (6, p. 31). (According to Sneed, these invariants express "*constraints*" on (the values of) f, i.e. cross-connexions among models of T[1]).

[1] It is this sense in which a function's being or not theoretical at a given theory is not an "empirical" matter (if "empirical" regards only what is or is not going on *in* (or *at*) a model).

Thus, if some relation A holds at any model x in (an appropriately ined class) B of models of T, say that A is a T-invariant only if it icerns the values of a function f of T, e.g. A has the form $f\ (d_1, \ldots d_s;\ \ldots a_p) = k\ (k \in \mathbb{R})$. Consequently, not any cross-connexion among dels of T which concerns some T-function f signals the T-theoretical ure of f; one needs only cross-connexions regarding the values of f. erefore, the only interesting thing about f with respect to theoreticity nes to determining, at any model and for any admissible arguments, elation f does or does not hold.

The formalization given by Balzer to classical particle mechanics *PM)* provides a suggestive illustration of this point. Function s, lzer argues, is *CPM*-nontheoretical, for, in general, $s_x \neq \alpha s_y$. However, is still possible to show that function s gives birth to certain cross-ations among *CPM*-models. Let B be a subtheory of *CPM* such that ind y are in B and $x_{-s}=y_{-s}$ i.e., x and y differ at most in their omponent. Then s_x and s_y are necessarily linked by

1.1. $s_x(p,\ t) = s_y(p,\ t) + vt + b$

some constants v and b. Balzer believes that the invariant expressed (1.1) is not significant to the theoretical/nontheoretical character of iction s at *CPM*. Now, by differentiating (1.1), one gets :

1.2. $\ddot{s}_x(p, t) = \ddot{s}_y(p, t)$

Assume for a moment that in reconstructing *CPM* one would take is a primitive notion, while s would be a derivative one. Then, according Balzer's criterion, (1.2) leads to $\ddot{s}$'s being *CPM*-theoretical.

The trouble with this view is that $\ddot{s}$ could not be primitive, for one ild not then be able to give conditions to fix, for any pair $(x,\ y)$ of *PM*-models, the constants v and b[2]. But it is of course possible to treat is an effective function of *CPM*. Then, by (1.2), $\ddot{s}$ needs to be *CPM*-eoretical, while s be *CPM*-nontheoretical. But, by a *purely mathematical* vice, i.e. by differentiating, one gets a theoretical function from a non-eoretical one. I think this conclusion does not fit very much our intui-ins ; together with the fact that Balzer gives no reason for his choice of e distinctive features of the theoreticity-involving-invariants, this shows at something must be wrong with our ways of thinking of what is for T-function to be given. My criticism will be focused on the assumption iat having the values of a function of a theory at any model of it and for iy admissible arguments is a sufficient condition for having that function.

These comments are intended to suggest that Balzer's criterion of ie theoretical is, somehow, much too narrow. In this sense, the modal iterion I proposed in (3) seems to be more general, for it relies on the xistence of *any* cross-relation ("constraint") on the theory's functions ; ccording to the view about theoreticity expressed in that paper a function

[2] However, the history of *classical mechanics* shows the long-ranging effort to effecti-ely determine, for each x, the values of v and b with respect to an absolute reference ystem x_0 (s_{x_0} $(p,\ t)$ denoting the absolute position of particle p at t). It is worth-noting iat Newton, who did admit of the existence of x_0, treated spatial rotations of the position unction s as relevant to the proof of the existence of an absolute reference system. There-ore, it would be interesting, perhaps, to study the consequences of including space rotations n Balzer's reconstruction of *CPM*.

f is T-theoretical iff its use at T essentially involves cross-relations among models of T.

It follows then that a function's property of being or not theoretical in a theory is relative to the kinds of cross-connexions one is ready to take into account in formulating his definition of the theoretical [3].

The argument to be developed below is not committed, however, to any view about the nature of these cross-connexions; it seems to me that it makes its point both when my view or Balzer's one are concerned [4].

II. **On the nature and strength of the argument.** The argument against the semantical definitions of the theoretical I wish to present below applies to all those attempts which :

2.1. Assume that a function f of a theory T is given iff its values at any model of T and for any admissible arguments are given. As I tried to argue in section I, this assumption is essentially involved in Balzer's approach to theoreticity. Let T be CPM; then, the argument asks e.g., that to completely determine function m is to determine at any model x of CPM the mass of each particle appearing at x.

2.2. Assume that the models of T are set-theoretical entities. This condition requires that each model of T be describable in set-theoretical terms.[5] My argument consists in showing that the class of T's models can be redescribed in a *non-standard* way.

It should be noted that the argument does not assume the possibility that T be a first-order theory, but that its models be set-theoretical constructs. The argument is closely related to Putnams's interpretation of the Löwenheim-Skolem theorem (5) : Putnam has persuasively argued that there are no systematic means to divorce intended from non-intended models of a theory. The present approach is committed to the claim that one cannot select intended interpretations of a theory's function, though all its properties (including its being or not theoretical at that theory)

[3] I think that the most important difference between Balzer's criterion of the theoretical and Sneed's one is this: constructing a function as T-theoretical is, according to Balzer, a by-product of the construction of class M_T of T's models. On the contrary, Sneed takes the theoretical character of a function as a precondition of the construction of M_T. Thus, while with Sneed the notion of *constraint* needs to be primitive, with Balzer it does not. It is for this reason why his criterion of the theoretical concerns nothing but the *values* of T's functions at any T-model, the *comparison* of these values being a *derivative matter*.

Note, however, that one could view my use of the alternativeness relation R (defined in section $\bar{V}$ below) as a means to show that the two strategies towards constraints are equivalent. For one can start either with constraints (identified with alternativeness relations like R) and define then what is or is not necessary at a certain model — as modal logicians use to do; or he can proceed as Balzer did and define then relations R (This sort of approach to constraints was developed in [3]).

[4] My definition does not lead, however, to inflationistic inventories of theoretical functions, e.g., to s's being CPM-theoretical. I avoid this unhappy result by denying that (1.2) really expresses a constraint at CPM and by arguing that it is a constraint on s only at PK (particle kinematics) of which CPM is a theoretization. I take then (1.2) to express a cross-connexion among models of PK.

[5] The essential claim is that the descriptions of the models of T and the cross-relations among them be translatable into a set-theoretical framework. That is why I think that the argument also holds against Sneed's restatement of his ideas by use of the theory of categories.

are preserved. But it diverges from Putnam in that it does not concern theories (which have models), but the class of the models of theories [6].

Balzer defines a model x of a theory T as a typified structure $x = (D_{1,x}, \ldots D_{k,x} ; A_{1,x}, \ldots A_{l,x} ; f_{1,x} \ldots f_{n,x})$, where sets $D_{r,x}$ (of "objects") are called "base sets", sets $A_{r,x}$ are called "auxiliary base sets" and $f_{i,x}$ are relations over $A_{r,x}$'s and $D_{r,x}$'s, which, of course, could be functions. At CPM, e.g., $k = 1$ and D_x is the set P of particles ; $l = 3$, the auxiliary base sets being an open interval $T \subseteq \mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^3$ (together with relations and operations on them), $n = m + 2$, CPM-functions being the position function s, the mass function m and forces f_i (compound forces) $i = 1, \ldots, m$).

Let T be a theory and let M_T be the class of its models. Define, for each model x, a set $G_x = \{g : g = (g_1, \ldots, g_k)$ and $g_r : D_{r,x} \to D_{r,x}$ $r = 1, \ldots, k)$ is a bijective function$\}$. If $x \in M_T$ and $g \in G_x$, define a structure $y = x^g$ by : $y = (D_{1,y}, \ldots D_{k,y} ; A_{1,y}, \ldots A_{l,y} ; f_{1,y}, \ldots f_{n,y})$, with $D_{r,y} = D_{r,x}$ $(r = 1, \ldots k)$, $A_{r,y} = A_{r,x}$ $(r = 1, \ldots l)$ and for each $i(i = 1, \ldots n)$, $f_{i,y}(d_1, \ldots d_s ; a_1, \ldots a_p) = f_{i,x}(g(d_1), \ldots g(d_s) ; a_1, \ldots a_p)$, where $a_1, \ldots a_p$ are in A_r 's and $g(d_u) = g_r(d_u)$, for $d_u \in D_{r,x}$ $u = 1, \ldots s)$.

Lemma 1. Each model x in M_T is a structure y^g, for some $g \in G_x$. The proof is simple, once we observe that G_x is a group, with the composition operation "o" defined by : $g \circ g' = (g_1 \circ g'_1, \ldots g_k \circ g'_k)$. Then there is one and only one g_0 in G_x so that $g_{0,r}(d) = d$ for each d in $D_{r,x}$ $(r = 1, \ldots k)$. Let x be in M_T ; then y^{g_0} is exactly x and therefore structure x has the form x^g for some g in G_x.

Lemma 2. If $x \in M_T$ and $g \in G_x$, then $x^g \in M_T$. The proof is left to the reader.

From lemma 2 it follows that if x^g is a model, then $(x^g)^{g'}$ is a model too. Indeed, by virtue of the definition of structures x^g and $(x^g)^{g'}$, it is possible to show that $(x^g)^{g'}$ is $x^{g \circ g'}$; but, G_x being a group, $g \circ g' = g'' \in G_x$ and thus $x^{g \circ g'}$ is a model $x^{g''}$.

Define set G by : $G = \{G_x : x \in M_T\}$. Let h be a function $G \to \cup G$ so that $h(G_x) \in G_x$ for each G_x in G ; let h_0 be the h-function of which it holds that $h_0(G_x) = g_0 \in G_x$, for each $G_x \in G$. (It is important to note that to make use of h-functions we assume of the axiom of choice). For any fixed function h of this sort, let H_h be a function from M_T to M_T so that $H_h(x) = x^{h(G_x)}$.

Theorem 1. H_h is bijective.

Proof : 1) If $x \neq x'$, then $H_h(x) \neq H_h(x')$. The consequent of this implication makes sense only if $G_x = G_{x'}$. Then there is a model y so that x is y^g and x' is $y^{g'}$, Now, from $H_h(x) = H_h(x')$ results that $y^{g \circ h(G_x)} = y^{g' \circ h(G_x)}$ and therefore $g \circ h(G_x) = g' \circ h(G_x)$, which holds only if $g = g'$. But in this case $y^g = y^{g'}$, i.e. $x = x'$, which contradicts with premiss $x \neq x'$.

[6] The argument shows that something must be wrong with the way we usually think of theories. It seems to me that the standard notion of a model-of-a-theory must be responsible for the counterintuitive import of the argument. It looks to me that a more general notion of model, grounded on appropriate semantical assumptions, is needed. But it is not the aim of the present paper to try to develop in some detail this idea.

2) H_h is on. Let $x \in M_T$; because G_x is a group, there is some g in G_x so that $g \circ h\ (G_x) = g_0$. But x^g is a model and consequently $H_h(x^g) = x^{g \circ h(G_x)} = x^{g_0} = x$.

III. Statement of the argument. My argument against Balzer's semantical definition of the theoretical is concerned with the way in which a function f_i of a theory T is thought of. Following Sneed, call f_i the i-th abstract function of T and say that $f_{i,x}$ is the concrete function subsumed under f_i and which appears at x (in the logicians' jargon, $f_{i,x}$ is the interpretation of f_i at x; note that Balzer himself made an explicit use of this spelling on page 133 of this paper (1)). By the first of the two assumptions set forth in section II above, knowing f_i is knowing all functions $f_{i,x}$, i.e. the values of $f_{i,x}$'s for all admissible arguments.

Then it is possible to identify (the intension of) f_i with a function F_i defined on M_T and having as values concrete functions: $F_i(x) = f_{i,x}$ ($f_{i,x}$ being the extension of f_i at x).

Now, assumption (2.1) is split into two parts. First, a function $f_{i,x}$ is held to be determined at model x iff the value of $f_{i,x}(d_1, \ldots d_s; a_1, \ldots \ldots a_p)$ is determined for any argument $(d_1, \ldots d_s; a_1, \ldots a_p)$; and second, a function f_i is held to be determined at theory T iff the value of $F_i(x)$ is determined for each x in M_T. Obviously, the theses involved in (2.1) share their logical form; however, the argument to be developed below concentrates mainly on the former one, while the latter will not be explored in much detail (though it is itself subject to the reiteration of the argument; see also for this issue note 14).

However, these two theses do not succeed in supporting Balzer's view on theoreticity: he takes (2.1) to involve a third one, namely that functions $f_{i,x}$ do uniquely determine function F_i; or, to put it in other words, functions $f_{i,x}$ (i.e., the extensions of f_i) are required to single out one *natural* (or: *intended*) way to construct function F_i, i.e. the intension of f_i at class M_T [7].

Now, it is possible to lay down the structure of my argument. It is argued that: 1) assumption (2.1) does not support Balzer's additional thesis; 2) assumption (2.1) brings about nonintuitive consequences, when used to Balzer's purposes; and 3) Balzer's view can be reconstructed on strong modal and possibilistic *de re* hypotheses.

Let us first observe that the definition of F_i could be restated as: $F_i(x) = f_{i,H_{h_0}(x)}$. The core of the first step of the argument is this: by substituting h_0 by h in the above expression, all the formal properties of F_i, including its T-theoretical/T-nontheoretical character, are preserved; we have no means to choose a single (natural) way to construct the i-th function of theory T. Indeed, let we start with the following (and, as proved below, equally reasonable) definition of F_i: $F_i\ (x) = f_{i,H_h(x)}$. By this definition, the extension of f_i at x is not $f_{i,x}$, but $f_{i,H_h(x)}$, i.e. $f_{i,y}$ (with $h\ (G_x) = g$ and $y = x^g$). Suppose f_i is, e.g., a function with values in $\mathbb{R}$. Balzer's intention with his criterion of the theoretical was this: the

[7] It is, perhaps, worth-noting that in this sense F_i is definable with respect to the class $\{f_{i,x}: x \in M_T\}$. However, I shall not be concerned in this paper with the use of Balzer's criterion in metatheory.

value of f_i at x for some admissible argument $(d_1, \ldots d_s;\ a_1, \ldots a_p)$ is k ($k \in \mathbb{R}$) iff

$$x \models f_{i,x}(d_1, \ldots d_s;\ a_1, \ldots a_p) = k$$

The first step of my argument amounts to constructing f_i in a quite different way by letting its value at x for some admissible argument $(d_1, \ldots d_s;\ a_1, \ldots a_p)$ be k iff

$$x \models f_{i,x}(g\,(d_1), \ldots g(d_s);\ a_1 \ldots a_p) = k$$

But, according to the definition of structure $y = x^g$, $f_{i,y}(d_1, \ldots d_s;\ a_1, \ldots a_p) = f_{i,x}(g(d_1), \ldots g\,(d_s);\ a_1, \ldots a_p)$. Then, the extension of f_i at x is function $f_{i,y}$. On the other hand, provided that $y = x^g$ is a model and that each H_h is bijective, it follows that to each model x there corresponds (*via* a certain function H_h) a uniquely determined model x^g. If h is h_0, then x^g is x, which shows that the present view is a generalization of Balzer's one. The first step of my argument is then this : it is possible to define F_i so that the value of f_i at x for any admissible argument be exactly the value which, according to the standard view, is the value of f_i at x^g for that argument.

Our talk about models received thus a non-standard interpretation to the effect that whenever we indend to deal with some model x, we actually deal with the model x^g. In this sense, my approach requires that the language we made use of to describe the models of a theory T receives a non-standard interpretation in that at least the *names* in it which stand for models of T do not refer to exactly those models we intend that they would refer to, but to other models of T.

By use of definition $F_i(x) = f_{i,H_{h_0}(x)} = f_{i,x}$, one yields then the n functions $f_1, \ldots f_n$ of T. Call them H_{h_0}-functions. If, on the other hand, one starts with definition $F_i(x) = f_{i,H_h(x)}$, then he yields n (possibly different) functions $f_1, \ldots f_n$ of T. Call them H_h-functions. I shall say that F_{i,H_h} is the H_h-function f_i; obviously, $F_{i,H_{h_0}}$ is the H_{h_0}-function f_i. I also say that f_i is H_h-T-theoretical iff the H_h-function f_i is T-theoretical.

Balzer's definition of theoretical functions simply generalizes to H_h-functions :

1) A set $B \subseteq M_T$ is M_T-H_h-f_i-invariant iff $(\forall xy)\,(x \in B \wedge x \simeq_i y \rightarrow y \in B)$

2) f_i is H_h-T-theoretical iff

a) $B \subseteq M_T$ is a species of structures ;

b) B is M_T-H_h-f_i-invariant ;

c) $(\forall xy)\,(x \in B\ \wedge\ y \in B \rightarrow f_{i,H_h(x)} \sim f_{i,H_h(y)})$

The main effort of this section is to prove the following theorem :

Theorem 2. f_i is H_h-T-theoretical iff it is H_{h_0}-T-theoretical. Proof. Let B be included in M_T and let it satisfy conditions (2a), (2b). If x, y are in B and $x \simeq_i y$, then $f_{j,H_h(x)} = f_{j,H_h(y)}$ for each $j = 1, \ldots n$, $j \neq i$. First, from $f_{j,H_h(x)} = f_{j,H_h(y)}$ results that $f_{j,x} = f_{j,y}$. Indeed, according to the definition of F_{j,H_h}, it holds that $f_{j,x}(g\,(d_1), \ldots g(d_s);\ a_1, \ldots a_p) =$

$= f_{j,y}(g(d_1), \ldots g(d_s); a_1, \ldots a_p)$, for every $d_1, \ldots d_s; a_1, \ldots a_p$. But each of the k component functions $g_r(r = 1, \ldots k)$ in g are bijective and therefore it holds that $f_{j,x}(d_1, \ldots d_s; a_1, \ldots a_p) = f_{j,y}(d_1, \ldots d_s; a_1, \ldots a_p)$, i.e. $f_{j,x} \equiv f_{j,y}$. Now, assume that f_i is H_h-T-theoretical; then, for every x, y in B, there is α so that $f_{i,H(x)} = \alpha f_{i,H(y)}$. Given the definition of $f_{i,H_h(\,)}$ it holds that $f_{i,x}(g(d_1), \ldots g(d_s); a_1, \ldots a_p) = f_{i,y}(g(d_1), \ldots g(d_s); a_1, \ldots a_p)$ for every $d_1, \ldots d_s; a_1, \ldots a_p$; then, it also holds that $f_{i,x}(d_1, \ldots d_s; a_1, \ldots a_p) = \alpha f_{i,y}(d_1, \ldots d_s; a_1, \ldots a_p)$, i.e. $f_{i,x} = \alpha f_{i,y}$, which completes the proof that f_i is H_{h_0}-T-theoretical. The other part of the theorem proves analogously.

Let T be, e.g., CPM. In this case, $k = 1$ and therefore $g = (g)$. For the sake of simplicity, I assume that the CPM-model x is a model $y^{g^{-1}}$ (G_x being a group, for each g in G_x there exists some g^{-1} such that $g \circ g^{-1} = g_0$). Then $x \simeq {}_2x'$ comes to $s_x(p, t) = s_{x'}(g(p), t)$; $f_{i,x}(p, t) = f_{i,x'}(g(p), t)$ $(i = 1, \ldots m)$. If there it holds that $(\forall)(\exists t)(\ddot{s}_x(p, t) \neq 0)$, then g being bijective, it also holds that $(\forall p)(\exists t)(\ddot{s}_{x'}(p, t) \neq 0)$.

From $m_x(p) \cdot \ddot{s}_x(p, t) = \sum_{i=1}^{m} f_{i,x}(p, t) = \sum_{i=1}^{m} f_{i,x'}((g(p), t) = m_{x'}(g(p)) \cdot \ddot{s}_{x'}(g(p), t)$ infer $m_x(p) = m_{x'}(g(p))$, i.e. $m_{H_h(x)} = m_{H_h(x')}$ and especially $m_{H_h(x)} \sim m_{H_h(x')}$, which results in m'' s being H_h-CPM-theoretical.

IV. **On the meaning of the argument.** I shall try to clarify in this section the second step of the argument by focusing on the example discussed at the end of section III. Under the H_h-interpretation of the functions appearing in CPM, if B is appropriately chosen, it is provable that if $x \simeq_2 x'$, then the extension of function m at x is equivalent (in the sense of (1, p. 133) with the extension of m at x', i.e. for each particle in the domain of x and also of x'.

4.1. $m_{H_h(x)}(p) = \alpha m_{H_h(x')}(p)$ with $\alpha = 1$.

However, though formally as good as H_{h_0}-functions, our H_h-functions fail to accomplish the intuitive intentions underlying Balzer's (and also Sneed's earlier) approach. Indeed, Balzer takes the existence of a class B of CPM-models which uniquely determines function m as defining condition for m 's being CPM-theoretical. His intuitions with this definition seem to be the following: let m be a map $P \to \mathbb{R}$, with $P = \bigcup_{x \in B} P_x$ so that $m = \bigcup_{x \in B} m_x$; then m is a function, i.e. if a particle p does appear both in the domain of x and also of y, and $m_x(p) = k$ and $m_y(p) = k'$, then $k = k'$ and, consequently, $m_x(p) = m_y(p)$.

The trouble with these intuitions springs once theorem 2 is taken into account. If assumption (2.1) holds, then, by theorem 2, m is both H_{h_0}-theoretical and also H_h-theoretical at CPM. Here I shall mainly concentrate on those aspects concerning the meaning of an expression like (4.1). The point is that equality is in total disagreement with the intuitive requirements assumed in Balzer's treatment of theoreticity. (4.1) is formally equivalent to

4.2. $m_x(p) = m_{x'}(g(p))$ for each particle p.

But it is of course possible that $m_x(p) = k$, while $m_{x'}(p) = m_{x'}(g(g^{-1})(p))) = m_x(g^{-1}(p) = m_x(p') = k'$, with $k \neq k'$. The counterintuitive result is thus that the H_h-function m was proved to be H_h-CPM-theoretical, though it attaches to the same particle p quite different values at different models of CPM.

Obviously, the first objection that comes to one's mind is that H_h-functions, as diverging from H_{h_0}-ones, are quite strange, obscure entities of which it is almost reasonable to delete with. H_{h_0}-functions, on the contrary, enjoy a logical or at least an epistemological priority over H_h-functions. Moreover, the T-functions we *intend* to make use of are H_{h_0}-functions.

However, the argument fails [8]: H_{h_0}-functions have no priority over H_h-ones. Let me sketch the proof, paying more attention to issues about logical priority. Assume a certain function H_h is fixed. Remember that, by theorem 1, H_h is bijective. Then it is possible to redescribe the definition of the H_{h_0}-function f_i of theory T by $F_i(H_h(x)) = f_{i,H_h(x)}$. The T-function just defined is, actually, $F_{i,H_{h_0}}$. But, by virtue of the definition of H_h-functions, we get $F_i(x) = f_{i,H_h(x)}$. This function is, actually, F_{i,H_h}. Therefore, it holds that $F_{i,H_h}(x) = F_{i,H_{h_0}}(H_h(x))$ and also $F_{i,H_h}(H_{h_0}(x)) = F_{i,H_{h_0}}(H_h(x))$, which shows the symmetry between H_{h_0}- and H_h-functions. It is thus possible to take H_h-functions as primitive and define H_{h_0}-functions analogously to the procedure we appealed to above when H-functions were defined. Indeed, let h^{-1} be a function of which it holds that: $h^{-1}(G_x) = (h(G_x)^{-1}$. Then function H_h-1 is a bijective H_h-function. Now, F_{i,H_h-1} which was defined with respect to F_{i,H_h} by $F_{i,H_h-1}(x) = F_{i,H_h}(H_h-1\ (x))$, is actually an H_{h_0}-function. Indeed, $F_{i,H_h}(H_h - 1(x)) = F_{i,H_h}(H_{h_0}(H_h - 1(x))) = F_{i,H_{h_0}}(H_h(H_h - 1\ (x))) = F_{i,H_{h_0}}(x)$. On the present approach, H_{h_0}-functions are strangely enough, for the extension of f_i at x is the extension of (what according to the H_h-function f_i is) the extension of f_i at model $H_h-1\ (x)$, i.e. $x^{g'}$, for some function $g' = g^{-1}$.[9]

Turning again to CPM, let m^+ be function m_{H_h} and let m be function $m_{H_{h_0}}$; assume that $h\ (G_x) = g$ and also that x has the form $x^{g'}$. Then, provided that m^+ is CPM-theoretical (for, as Balzer proves, m is CPM-theoretical and theorem 2 applies), $m^+_x(p) = m^+_y(p)$ holds. Now this expression holds [10] iff it also holds that $m_x(p) = m_x(g(p))$. But the two equations differ in their logical form, for while the former concerns the values at two models of the mass function for some particle p, the latter one provides

[8] Balzer's approach to theoreticity is essentially committed to the notion of Ψ-transport [1, p. 131]. A ψ-transport involves both: 1) bijections on sets $D_{r,x}$ $(r = 1, \dots k)$; and 2) the corresponding "transports" of the (values of) functions $f_{i,x}$ $(i = 1, \dots n)$. My argument relies on a sharp split of the two aspects involved in the notion of ψ-transport. The transformations considered are given simply by bijections Ψ_r on sets $D_{r,x}$, while functions f_i are not transported, they remain unchanged. It is for this reason why, as I believe, Balzer's approach fails if confronted with the H_h-functions argument.

[9] If some criterion were laid down with a view to divorcing H_{h_0}-functions from H_h-ones, it would still be possible to think of it as of a H_h-criterion. Therefore, it could not make its point.

[10] Note that the present use of expressions like $m^+_x(p) = m^+_y(p)$ is not committed to the assumption that they hold at some model. This view sharply divges from the one adopted in the next section. However, I do not aim at clarifying in the peresent paper the semantical assumptions involved in these two views about models (see also note 6 on this issue).

a comparison of the values of the mass function at two models for two different particles, namely p and $p' = g(p)$.

It seems then reasonable to claim that in, e.g., "$m_y^+ g'(p) = k$", the argument of the H_h-function m^+ is not p, but $g\,(g'(p)) = p'$, while in "$m_z^+ g''(p) = k$" its argument is $g\,(g''(p)) = p''$ (and it is of course possible that $p' \neq p''$).

I believe this points a very important issue concerning the status at a theory T, e.g. CPM, of the elements in sets D_i. Balzer looks to rely on the following view (it seems to me that it is consistent with our intuitions about particles) : let $P = \bigcup_{x \in M_{CPM}} P_x$. A particle p is identified with a function $\mathbf{p} : M_{CPM} \rightarrow P$ such that for all x, $\mathbf{p}(x)$ is an element, say p, in P. Thus, $\mathbf{p}$ selects at every model x of CPM the same entity p[11]. It is for this reason that p could then be substituted in all contexts by p. (To put it in other words, p is the intension of the constant "p" and p is, at any model, the extension of "p"; second, "p" is *rigid*, i.e. its intension is a constant function).

However, once we rely on the H_h-functions argument, a quite different way to think of particles is needed. A particle is, in this view, a partial function $\mathbf{p} : M_{CPM} \rightarrow P$ of which it holds that : 1) if $\mathbf{p}(x)$ is defined, then $\mathbf{p}(x) \in P_x$; 2) $\mathbf{p}(x^g) = g\,(\mathbf{p}(x)$; 3) if $\mathbf{p}(x)$ is not defined and $G_x = G_y$, then $\mathbf{p}(y)$ is not defined [12]. On this view, "$\mathbf{p}$" is not rigid anymore.

This case faces a close analogy with the cross-identification puzzle in modal logic : are there cross-world or world-bound individuals? Balzer seems to admit of the same individual's (e.g. particle) being the inhabitant of more than one model of the theory. He also assumes that there are systematic means to identify it in each model in which it exists. As opposed to Balzer's approach, the H_h-functions argument is not committed to these two assumptions (and especially to the second one). Indeed, given a class of T-models, one needs not to select the same individual at different elements of it; rather it is required, e.g., that each individual be uniquely correlated at any other T-model, say y, with some (perhaps different) individual. Balzer's view, as described above, seems therefore to entail stronger *de re* commitments. However, their nature is not clear so far. I shall have more to say on this issue at the end of the final section of the paper, devoted entirely to modal topics in theory reconstruction.

Remark. Assume that $m_x^+(p) = m_y^+(p)$ is interpreted, under a Balzer-type approach to particles, as asserting something about $\mathbf{p}$, i.e. p. Then it would mean : $m_x(p) = m_y(p)$. The trouble with this suggestion is that it does not preserve all the properties of the CPM-mass function. Indeed, though $m_{H_{h_0}}$ was proved to be CPM-theoretical, m_{H_h} needs not share (under the assumed interpretation) this feature.

[11] If $\mathbf{p}(x) \notin P_x$, then p does bot exist at x (but it is still the extension of $\mathbf{p}$ at x).

[12] A more general approach to this problem is the following: let the elements of the base-sets of a model x of T be reconstructed as functions in the following way. If there is some $x \in M_T$ so that d_s is in $D_{i,x}$, then d_s is identified with a function $\mathbf{d}_s : M_T \rightarrow \bigcup_{x \in M_T} D_{i,x}$. Obviously, as suggested above on the CPM-example, there are at least two different ways to define $\mathbf{d}_s$. Let $\mathbf{D}_i$ be a set of functions $\mathbf{d}_s$; then a model x of T can be reconstructed by: $x = (\mathbf{D}_{1,x}, \ldots \mathbf{D}_{k,x}; A_{1,x} \ldots A_{1,x}; f_{1,x}, \ldots f_{n,x})$.

V. Modal axiomatizations and the Ramseyfication of theories. Balzer's criterion of the theoretical does not then work, for it cannot divorce intended from non-intended interpretations of a theory's functions. And yet—as I shall try to show below—it can have a good use in an appropriate understanding of what is for a function f_i of a theory T to be T-theoretical. Now have a moment's reflection to the very nature of the contexts brought about above and of which I held that they count against the semantical definitions of the theoretical. It seems to me that they all share an intensional character. This was apparent with the issues about the status at T of the elements of sets D_i; but the use of the H_h-functions also appealed to intensional contexts. Indeed, my argument implies that one can have different — and equally good — ***intensions*** of the theory's functions, of which some are not intended. To put it in another way : the semantical definitions of the theoretical cannot accommodate intensional contexts of theory use.

It is this reason why I shall take into account the issue of modal axiomatizations of theories as an attempt to dissolve this sort of argument against the semantical approach to theoreticity.

Let T be a theory and let A_T be an axiomatization of it in a classical first-order language L. Let L be enriched to a language Lm by adding to it the necessity operator N. The axiomatization of T in Lm shall be then *modal*, i.e. a Lm-axiomatization. The underlying modal logic is assumed to be the Brouwerian system (B)[13].

Let $J_{T,t} = \{f_i\}_{\substack{i=1,\dots u \\ u \leq n}}$ be a family of T-functions.

I shall say that mA_T is a Lm-axiomatization of T in the language Lm if the specific T-axiom

$$\mathbf{T} : (\exists f'_1, \dots \exists f'_u)\,(N(f_1 = f'_1) \wedge \dots N(f_u = f'_u) \wedge NA_T(f_1/f'_1, \dots f_u/f'_u))$$

is added to the axioms of functional logic and modal system B. Here $f_i = f'_i$ $(i = 1, \dots u)$ is short for $(\exists \alpha \,\forall d_1, \dots \forall d_s \,\forall a_1, \dots \forall a_p)$ $(f_i(d_1 \dots \dots d_s;\ a_1, \dots a_p) = \alpha\,(f'_i\,(d_1, \dots d_s;\ a_1, \dots a_p))$ and $A_T(f_1/f'_1 \dots f_u/f'_u$ is the result of substituting in A_T f_i-functions by f'_i-ones $(i = 1 \dots u)$.

5.1. Definition. f_i is T-theoretical iff $f_i \in J_{T,t}$ and mA_S is a consistent Lm-axiomatization of T, for some set $J_{T,t}$.

Let, e.g., A_{CPM} be the axiomatization of R. Montague (4) of CPM. Take $J_{CPM,t}$ be the set $\{m, f\}$. (I shall not consider there more special issues about the formal structure of f, or of the family of functions which could be used to replace it). Now define on M_{CPM} (the class of admissible interpretations of mA_{CPM}) a relationship R by

$$R\,(x,\ y) \text{ iff } \{X : x \models N\,X\} \subseteq \{X\ : y \models X\}$$

If the underlying modal system is B, then R is reflexive and symmetrical; and if B is strengthened to S5, R comes to an equivalence relationship on M_{CPM}.

[13] I believe, this choice is supported (besides some other reasons I shall not take into account here) by Balzer's use of (partially overlapping) subtheories of T in which a function f_i is uniquely determined.

CPM is, obviously, true at any model x in M_{CPM}; therefore, "$N\ (m'_y = m)$" is true at x for some function m'_x. If $R\ (x, y)$ holds, then "$m'_x = m$" is true at y. But, provided that y itself is a *CPM*-model, "$N\ (m'_y = m)$" is true at y for some function m'_y and hence, R being reflexive, "$m'_y = m$" is true at y. As a result, "$m'_x = m'_y$" is true at y. It is very important to notice that $m'_x(p) = m(p)$ holds both at x and also at y and that at y it also holds that $m'_x(p) = m'_y(p)$. But this does not exclude the possibility that "$m(p) = k_1$" be true at x, while "$m(p) = k_2$" be true at y, with $k_1 \neq k_2$, i.e. that $m'_x(p) = k_1$ holds at x and $m'_x(p) = (= m'_y(p)) = k_2$ holds at y. This shows that the Lm-axiomatization of *CPM*, as formulated above, admits of conceiving the *CPM*-functions as H_h-ones.

The proof that m is *CPM*-theoretical reduces to the proof that mA_{CPM} is consistent (it is this sense in which I claim that definition 5.1. provides *a logical criterion of the theoretical*). Now, a consistency proof of a Lm-axiomatization of theory T amounts to proving that there is a "modal structure" (M_T, R), with M_T and R defined as above.

The present approach to theoreticity is subject, however, to a fierce criticism. Indeed, "being T-theoretical" seems to be relativized to "being T-theoretical with respect to a certain modal axiomatization of T". Then, the argument goes on, the proof that a function f_i is T-theoretical is not required to involve the unicity of the choice of set $J_{T,t}$.

It looks to me that the above criticism fails. Let A_{CPM} be Montague's axiomatization of *CPM*. Then there is one and only one consistent modal axiom **CPM**, if A_{CPM} is consistent. One can prove, e.g., that at *CPM* the position function s cannot be added to the set $J_{CPM,t} = \{m, f\}$. But it is also possible to show that here is no way to construct $J_{CPM,t}$, so that s would be a member of it.

Let us assume that A_{CPM} is consistent. According to the above definition of theoreticity, if m is *CPM*-theoretical, then there is a consistent axiom **CPM**, i.e. if m is *CPM*-theoretical, then A_{CPM} entails that **CPM** is consistent. On the other hand, if m is *CPM*-theoretical, then *CPM* entails that $(\exists\, m')\ NA_{CPM}(m/m')$ and further $(\exists\, m')\ A_{CPM}(m/m')$. Now A_{CPM} and $(\exists\, m')\ A_{CPM}(m/m')$ are deductively equivalent; therefore, if $(\exists m')A_{CPM}(m/m')$ is consistent, then A_{CPM} is consistent too. Consequently, if m is *CPM*-theoretical, it follows that if **CPM** is consistent, then A_{CPM} is consistent. We conclude that m's being *CPM*-theoretical presupposes that **CPM** and A_{CPM} are equivalent with respect to the consistency condition.

A very powerful means to deal with the uniqueness condition concerning the choice of set $J_{T,t}$ (in particular, the choice of $J_{CPM,t}$) is fortunately supplied by Balzer's approach to theoreticity. Let us apply it to *CPM*. Suppose that function s is *CPM*-theoretical, i.e. the specific axiom **CPM** of mA_{CPM} entails that

$$(1) \qquad (\exists\, s')\ (N(s' = s) \wedge N\ A_{CPM}(s/s',\ m/m',\ f/f'))$$

Analogously to the proof I carried out above with respect to the mass function m, if x and y are CPM-models of which it holds that $R(x, y)$, then

(2) $$s'_x = s'_y$$

is true at y. On the other hand,

(3) $$\ddot{s}'_x(p, t) \cdot m'_x(p) = \sum_i^n f'_x(p, t, i)$$

is necessary at x and therefore true at y. But y being a model, it holds at y that

(4) $$\ddot{s}'_y(p, t) \cdot m'_y(p) = \sum_i^n f'_y(p, t, i)$$

Let y be such that $m'_x = m'_y$, $f'_x = f'_y$ (it is at this moment that the demonstration appeals to Balzer). This assumption does not contradict the assumption that $R(x, y)$. From (3) and (4) it follows :

(5) $$\ddot{s}'_x(p, t) = \ddot{s}'_y(p, t)$$

By integrating twice we get

(6) $$s'_x(p, t) = s'_y(p, t) + tv + b$$

But, if v and b are suitably chosen, (6) together with (2) yield a contradiction.

I think that, given the results of the present section, Balzer's approach to theoreticity could be better interpreted not as an attempt to offer a definition of "term t of a theory T being T-theoretical", but rather as a means to show that this property of a term is not relative to the choice of a certain axiomatization of T.

In the remainder of this paper I shall examine in more detail the logical structure of axiom **T**. **T** shows a sort of analogy with the Ramsey-sentence of a theory, in that it involves quantification over the theory's functions. In a sense, it reinforces the bearing of the Ramsey-sentence of theory T on the dichotomy of theoretical from nontheoretical functions of T. But, while on the standard account constructing the Ramsey-sentence presupposes the dichotomy, on the present one **T** does provide a (logical) *criterion* for taking some of T's functions be theoretical at it. However, in **T** T-functions f_i still do occur. Quantifiers range over functions necessarily equivalent to f's. $\exists f_i^*$-type quantification was used in **T** with a view to handle the (intensional) contexts falling under the scope of the necessity operator (this formal trick is due to R. Montague).

Balzer's approach seems to require a stronger *possibilistic* quantification over function-type entities and also over the individual variables (including both the specific and also the non-specific variables of **T**). Indeed, the trouble about the status of individuals at a theory (which I have mentioned in the previous section) and also the consistency of *Lm*-axiomatizations with the use of H_h-functions of the theory do asume of actualistic quantifications.

To do this, let us first enrich *Lm* to a *modal and possibilistic* language *Lmp* by adding to it the possibilistic quantifiers Σ and π besides the actualistic ones. Then, we turn the modal axiomatization **T** of *T* into a modal and possibilistic one $\mathbf{T}_B$ (see (2) for a detailed account of this issue) as follows : all actualistic $\exists f_i$-type quantifiers in **T** are replaced by possibilistic ones ; and second, for each f_i ($i=1, \dots u$), the expression "$N(f_i = f_i')$" is replaced by "$(\Sigma\alpha\pi d_1, \dots \pi d_s\pi a_1, \dots \pi a_p\pi k)$ $N((d_1, \dots d_s; a_1, \dots a_p; k) \in f_i \equiv (d_1, \dots d_s; a_1, \dots a_p; \alpha \cdot k) \in f_i')$".

I take $\mathbf{T}_B$ to be in good agreement with the naturalistically minded philosopher's view on a theory's domain and also with his conviction that H_h-functions and H_{h_0}-functions could not be on the same par. The main formal advantage of formulating $\mathbf{T}_B$ is that it clearly shows the nature and the strength of this philosopher's *de re* commitments.

As far as our main purpose is to find out a criterion for a function's being theoretical at a theory, I believe that Ockhman's razor — be committed to *de re* claims only if necessary ! — should be taken as a most important means of appraising alternative approaches to theoreticity. That is why I do not agree with the use of $\mathbf{T}_B$-axioms [14]; on the other hand, T-axioms are, as I tried to show above, much too weak. The appropriate solution to this dilemma seems to me to lie in semantics rather than in methodology. But it was not the aim of the present paper to develop it (one could, e.g., rely on a Putnamian position to avoid the H_h-functions argument); for I have simply tried to provide an argument *against* the semantical definitions of the theoretical.

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[14] It is necessary to notice that the argument against Balzer's definition of the theoretical does not dissolve with establishing the nature and the strength of his *de re* claims. The argument could be reiterated somehow as follows: call (M_T, R) a model-structure, where R is a relationship defined (as indicated above) on M_T. Now, if one provides a semantical frame for modal (and possibilistic) axiomatizations and if the model-structures he brings about are set-theoretical constructs, then it is still possible to redescribe the class of T's model-structures in a non-standard (non-intended) manner a.s.o.