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MODAL LOGIC REPRESENTATIONS FOR SOME SUBSYSTEMS OF ASSERTORIC SYLLOGISTIC

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Consider the following Lukasewiczian axiomatization of assertoric syllogistic :

A1. AaA	A5. $AaB \rightarrow \overline{AoB}$
A2. AiA	A6. $\overline{AaB} \rightarrow AoB$
A3. $AaB.BaC \rightarrow AaC$	A7. $AiB \rightarrow \overline{AeB}$
A4. $BaC.BiA \rightarrow AiC$	A8. $\overline{AiB} \rightarrow AeB$

Let **A** be the set of axioms A1-A8. The aim of this paper is to provide a semantics for certain subsets of **A** and to discuss on this basis the question of representing assertoric syllogistic in modal propositional logic.

I

An **A**-model is a triple $m = (K_m, J_m, v_m)$, where : 1.1 K_m is an algebra of classes on the non-empty set K_m ; 1.2.J_m is a subset of $P(K_m)$ (the power-set of K_m) satisfying the following conditions :

1.2.1. $\cup J_m = K_m$

1.2.2. $\emptyset \notin J_m$

1.2.3. If A, B are in J_m, then either $A \cap B = \emptyset$, or there is some C in J_m so that $C \subseteq A \cap B$

1.2.4. $J_m \neq \emptyset$

By 1.2.4. one is sure that the model is not trivial; 1.2.1 is a covering condition to the effect that the conceptual frame is complete (with respect to the individuals in the domain K_m of m); by 1.2.3. ecthesis is allowed. Aristotle uses it in the proof of *Darapti* (*Pr. An.*, I, 6, 28a) and suggests it is also applicable to *Disamis* and *Datisi* (*Pr. An.*, I, 6, 28b).

Finally, condition 1.2.2 conveys the usual requirement that assertoric syllogistic applies to nonempty terms.

1.3. v_m is a function satisfying conditions :

1.3.1. $v_m(A) \in J_m$ for each term A.

I shall write A_m for $v_m(A)$.

1.3.2. For any syllogistical expression X, $v_m(X)=1$ or $v_m(X) = 0$;

1.3.3. $v_m(-X) = 1$ iff $v_m(X) = 0$;

1.3.4. $v_m(X \cdot Y) = 1$ iff $v_m(X) = v_m(Y) = 1$;

1.3.5. $v_m(X \vee X) = 0$ iff $v_m(X) = v_m(X) = 0$

(here "1" denotes truth and "0" denotes falsehood).

Note. Usually, models m for $\mathbf{A}$ are not conceived of as entities involving a set J of possible extensions at m of syllogistical terms. It seems to me that the enrichement of m from (K_m, v_m) to (K_m, J_m, v_m) I adopted in this paper is to be preferred, for, first, it simplifies and systematizes some sorts of semantical conditions for assertoric syllogistic (especially the requirement that syllogistic should deal with nonempty terms). Second, as shown in section IV below, some developments appeal to "constraints" on the set of possible extensions of terms at different models, which can easily be represented by use of set J. Third, I believe there are also some exegetic grounds for taking J as a primitive notion in syllogistic. Aristotle claimed (*Post. An.*, II, 2, 89b – 90a ; 8, 93b) that the choice of middle terms is presupposed by any demonstration. Another argument lies in his discussion in *Pr. An.* on the choice of the middle (I, 41 – 44) and the characteristic features of the middle (I, 13, 32b). However, I shall not focus here on defending this point.

An $\mathbf{A}$ -model structure is a set F of $\mathbf{A}$-models. I shall say that a syllogistical expression X is F-valid iff for any m in F, $v_m(X) = 1$; and that X is $\mathbf{A}$-valid iff for any F, X is F-valid.

However, the definition of function v must be completed with satisfaction requirements for atomic expressions of syllogistic like AaB, AeB, AiB, AoB. Yet they can diverge when different subsets of $\mathbf{A}$ are taken into account.

An AS (assertoric syllogistic)-model structure is an $\mathbf{A}$-model structure of which it holds :

1.3.6. $v_m(AaB) = 1$ iff $A_m \subseteq B_m$;

1.3.7. $v_m(AoB) = 1$ iff $A_m \cap -B_m \neq \emptyset$;

1.3.8. $v_m(AeB) = 1$ iff $A_m \cap B_m = \emptyset$;

1.3.9. $v_m(AiB) = 1$ iff $A_m \cap B_m \neq \emptyset$.

Theorem 1. X is AS-valid iff it is a theorem of assertoric syllogistic.

I shall sketch the proof of the necessity part of this theorem : if X is AS-valid, then it is a theorem in $\mathbf{A}$. Assume that X is not a theorem ; then it must have a counter-model, i.e. $-X$ must have a model. To show that, start with the consistent set $\{-X\}$. Let H be the set of all syllogistic terms occurring in X. Then, extend $\{-X\}$, with respect to H, to a maximal consistent set $\mathbf{X}$ of syllogistical expressions. To construct the canonical model m, $\mathbf{X}$ shall be extended to another consistent set $\mathbf{X}'$ as follows : 1) if AiB is in $\mathbf{X}$, then add to $\mathbf{X}$ expressions $C'aA$ and $C'aB$, with C' a term not occurring in H; 2) if $A'iB'$ and $A''iB''$ are in **X**, with either $A' \neq A''$ or $B' \neq B''$, then $C' \neq C''$; 3) for any two terms C', C'' not occurring in H add to **X** expressions $C'eC''$, $C''eC'$. Let H' be the set of the new terms C', C'' ...

Now the definition of m goes as follows: K_m is the union of sets H and H'; J_m is a set of sets of terms in K_m and T is in J_m iff there is some term A in K_m so that T is $v_m(A)$. Let A be in K_m; then put $B \in v_m(A)$ iff BaA is in **X'**.

From A1. AaA infer $A \in v_m(A)$ for any A and therefore 1.2.2. holds; as a corollary, condition 1.2.1 is satisfied too. Assume now that $v_m(A) \cap v_m(B) \neq \emptyset$. Then there is some D in K_m and $v_m(D)$ is contained in $v_m(A) \cap v_m(B)$. Indeed, if $v_m(A) \cap v_m(B)$ is not empty, then there is some D belonging to it; but DaA and DaB are in **X'**. Let C be in $v_m(D)$, i.e. CaD is in X'. Then, by A3' CaA and CaB are in **X'**; therefore, $C \in v_m(A)$ and $C \in v_m(B)$, i.e. $C \in v_m(A) \cap V_m(B)$. Consequently, $v_m(D)$ is included in $v_m(A) \cap v_m(B)$.

Function v_m fulfils conditions 1.3.6. — 1.3.9. Consider, e.g., condition 1.3.9. If $v_m(A) \cap v_m(B) \neq \emptyset$, then there is some D so that D is in $v_m(A)$ and also in $v_m(B)$. Then DaA and DaB are in **X'** and, by A4, AiB is in **X'**. If, on the other hand, AiB is in **X'**, then there must be some C in K_m so that CaA and CaB are in **X'**. But it means that C is in $v_m(A)$ and also in $v_m(B)$, i.e. $v_m(A) \cap v_m(B)$ is not an empty set.

II

In this section I shall define Ai-model structures. They fulfil the important property that all axioms Aj in **A** except for Ai are Ai-valid. Consequently, Ai-model structures provide a means to carry out independence results in assertoric syllogistic. The underlying intuitive idea is to interpret modally — as a sort of necessity or of possibility — some of the syllogistical relations.

Ai-model structures are obtained by adding to the definition of v_m certain sets of satisfaction conditions for atomic expressions. First, I shall describe A3-A8 model structures. They share the property that no further conditions (or cross-conditions — i.e. conditions involving connexions among different models in F) on K_m and J_m are required.

A3-model structures

1.3.6. $v_m(AaB) = 1$ iff there is some m' in F so that $A_{m'} \subseteq B_{m'}$;

1.3.7. $v_m(AoB) = 1$ iff for any m' in F, $A_{m'} \cap -B_{m'} \neq \emptyset$;

1.3.8. $v_m(AeB) = 1$ iff $A_m \cup -A_m \subseteq A_m \cap -A_m$;

1.3.9. $v_m(AiB) = 1$ iff $A_m \cap -A_m \subseteq A_m \cup -A_m$.

Note. In virtue of 1.2.2, one is sure that for any A, B, m, $v_m(AeB) = 0$, $v_m(AiB) = 1$.

A4-model structures

1.3.6. $v_m(AaB) = 1$ iff $A_m \subseteq B_m$;

1.3.7. $v_m(AoB) = 1$ iff $A_m \cap - B_m = \emptyset$:

1.3.8. $v_m(AeB) = 1$ iff there is some m' in F so that $A_{m'} \cap B_{m'} = \emptyset$;
1.3.9. $v_m(AiB) = 1$ iff for any m' in F, $A_{m'} \cap B_{m'} \neq \emptyset$.

A5-model structures

1.3.6. $v_m(AaB) = 1$ iff $A_m \subseteq B_m$;
1.3.8. $v_m(AeB) = 1$ iff $A_m \cap B_m = \emptyset$;
1.3.7. $v_m(AoB) = 1$ iff there is some m' in F so that $A_{m'} \cap -B_{m'} \neq \emptyset$ (= there is some m' in F so that $v_{m'}(AaB) = 0$);
1.3.9. $v_m(AiB) = 1$ iff $A_m \cap B_m \neq \emptyset$.
As for A6—A8 structures we have
1.3.6, $v_m(AaB) = 1$ iff $A_m \subseteq B_m$;
1.3.9. $v_m(AiB) = 1$ iff $A_m \cap B_m \neq \emptyset$.
and

A6-model structures

1.3.7. $v_m(AoB) = 1$ iff for any m' in F, $A_{m'} \cap - B_{m'} \neq \emptyset$;
1.3.8. $v_m(AeB) = 1$ iff $A_m \cap B_m = \emptyset$.

A7-model structures

1.3.7. $v_m(AoB) = 1$ iff $A_m \cap -B_m \neq \emptyset$;
1.3.8. $v_m(AeB) = 1$ iff for any m' in F, $A_{m'} \cap B_{m'} = \emptyset$.

A8-model structures

1.3.7. $v_m(AoB) = 1$ iff $A_m \cap - B_m \neq \emptyset$;
1.3.8. $v_m(AeB) = 1$ iff there is some m' in F so that $A_{m'} \cap B_{m'} = \emptyset$.
It is easily provable that:

Theorem 2. (i) For any axiom Ai, with $3 \leqslant i \leqslant 8$, Ai is not Ai-valid;
(ii) For any i, j, $3 \leqslant i \leqslant 8$, $i \neq j$, Aj is Ai-valid.

III

In his book, *Axiomatizări și modele ale sistemelor silogistice*, Ed. Academiei, București, 1975, p. 36 — 38, S. Vieru developed a procedure to prove independence results in syllogistic by representing each term as a pair of terms. For example, in the case of A7, S. Vieru gives the following representation:

$AaB \rightarrow A'aB'$
$AiB \rightarrow A'iB'$
$AeB \rightarrow A'eB'. A''eB''$
$AoB \rightarrow A'oB'$

He shows that by replacing AaB, AiB... according to the above representation in A1—A8, one obtains theorems of assertoric syllogistic too, excepting the case of A7. Indeed, A7 turns to

A7$'\cdot \overline{A'iB'} \rightarrow A'eB' \cdot A''eB''$

which is not a consequence of A1—A8.

However, I wish to show that Vieru's approach is grounded semantically. To prove, e.g., that A7 is not A7-valid is suffices to construct a model structure $F = \{m', m''\}$ enjoying the property that A7 is not F-valid. Now, for A7-model structures, conditions 1.3.6—9 reduce, in the case of F and m', to :

1.3.6. $v'(AaB) = 1$ iff $A' \subseteq B'$;
1.3.7. $v'\ (AoB) = 1$ iff $A' \cap - B' \neq \emptyset$;
1.3.8. $v'\ (AeB) = 1$ iff $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$;
1.3.9. $v'\ (AiB) = 1$ iff $A' \cap B' \neq \emptyset$.

If there are A and B so that $A' \cap B' = \emptyset$, while $A'' \cap B'' \neq \emptyset$, then $v'\ (A7') = 0$.

S. Vieru showed that his procedure might be handled so that to reject Lukasiewicz's claim that in order to prove the independence of A3, two-valued propositional logic does not suffice and that one should appeal to some kind of many-valued logic. To do that Vieru considers a model of syllogistic in standard two-valued propositional logic. The standard correspondence

$AaB \to A \to B$
$AiB \to A \cdot B$
$AeB \to A \to -B$
$AoB \to A \cdot -B$

invalidates A2. But, as Vieru proved, a wider class of propositional models is available where each term A be represented as a pair of variables in two-valued logic. As for A3 he gives the following model :

$AaB \to (A' \to B') \vee (A'' \to B'')$
$AoB \to (A' \cdot -B') \cdot (A'' \cdot - B'')$
$AeB \to (A' \cdot - A') \vee (A'' \cdot A'')$
$AiB \to (A' \vee - A) \cdot (A'' \vee - A'')$

I wish to restate Vieru's results in the more general framework sketched above. Let me identify the members of the power-set of $K(=P(K))$ with truth-values in a card$/P(K)/$— valued logic. Suppose now that $K = \{b, b'\}$; consequently, we get a 4-valued logic, with disjunction corresponding to union, negation to complementation a.s.o. on $P(K)$. By use of the representation $\{b, b'\}$ — 1, $\{b\} \to 2$, $\{b'\} \to 3$, $\emptyset \to 4$, we easily obtain matrixes :

p	$-p$
1	4
2	3
3	2
4	1

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

and also

$\vee$	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

Let $F = \{m', m''\}$ be an A3-model structure. Definitions 1.3.6—9 come to

1.3.6. $v(AaB) = 1$ iff $A' \subseteq B'$ or $A'' \subseteq B''$;

1.3.7. $v(AoB) = 1$ iff $A' \cap -B' \neq \emptyset$ or $A'' \cap -B'' \neq \emptyset$;
1.3.8. $v(AeB) = 0$;
1.3.9. $v(AiB) = 1$,
v being either v', or v''.

Now, if we interpret A', A'', B', B'' as propositional variables, and $\subseteq$ $\cap$,$-$, respectively as implication, conjunction, negation, then we meet again Vieru's model.

Note that, according to 1.2.2, propositional variables could only be 2 or 3; however, Boolean compounds range over 1 and 4 too; indeed, e.g., $2 \rightarrow 2 = 1$, $3.2. = 4$.

Syllogistical relations can be represented as logical operators; as one can easily see, o behaves in the present case like a sort of necessity, while a behaves like a sort of possibility. Let me define :

p	$a(p)$	$o(p)$	$e(p)$	$i(p)$
1	1	1	4	1
2	1	4	4	1
3	1	4	4	1
4	4	4	4	1

For syllogistical purposes, we need but a small part of the four-valued modal logic just defined; we only need expressions like $a(A \rightarrow B)$— corresponding to AaB —, $o(A.-B)$ — corresponding to AoB—, $e(A \rightarrow -B)$ — corresponding to AeB—, and $i(A.B)$— corresponding to AiB[1].

These all complete the semantical counterpart of Vieru's model.

IV

In this section I turn to A1-and A2-model structures; I also try to discuss some of the intuitive intentions which lie behind the formalism.

Let $F = \{m', m''\}$ be an A1-model structure. Vieru's proposal, in this case is, e.g., that

1.3.6. $v'(AaB) = 1$ iff $A' \subseteq B'$ and $A'' \subseteq B'$.

What makes the A1-model structures differ from all structures considered above is that they involve a relationship like $A'' \subseteq B'$. But remember that A'' is $v''(A)$ and B' is $v'(B)$, i.e. a cross-connexion between the models in F is required.

To be sure that relations like $A'' \subseteq B'$ make always sense, we have to add this constraint on K :

1.1.1. $K_{m'} = K_{m''}$ for any m', m'' in F.

A1-model structures

1.3.6. $v_m(AaB) = 1$ iff for any m' in F, $A_{m'} \subseteq B_m$;
1.3.7. $v_m(AoB) = 1$ iff there is some m' in F so that $A_{m'} \cap -B_m \neq \emptyset$;
1.3.8. $v_m(AeB) = 1$ iff $A_{m'} \cap B_m = \emptyset$;
1.3.9. $v_m(AiB) = 1$ iff $A_{m'} \cap B_m \neq \emptyset$.

A2-model structures

1.3.6. $v_m(AaB) = 1$ iff for any m' in F, $A_{m'} \subseteq B_m$,
1.3.7. $v_m(AoB) = 1$ iff there is some m' in F so that $A_{m'} \cap -B_{m'} \neq$
$\neq \emptyset$;

[1] The modal logic representation is then this: at S5 take $\Diamond(p \rightarrow q)$ correspond to AaB, $\Box(p \cdot -q)$ to correspond to AoB a.s.o.

1.3.8. $v_m(AeB) = 1$ iff there is some m' in F so that either $A_{m'} \cap B_m = \emptyset$, or $A_m \cap B_{m'} \neq \emptyset$;

1.3.9. $v_m(AiB) = 1$ iff for any m' in F, $A_{m'} \cap B_m \neq \emptyset$ and $A_m \cap B_{m'} \neq \emptyset$.

With respect to A2-model structures, by use of, say, $F = \{m', m''\}$ we get the following Vieru countermodel of A2 (*Axiomatizări și modele . . .*, p. 38):

$AaB \rightarrow A'aB' \cdot A''aB''$

$AiB \rightarrow A'iB' \cdot A'iB'' \cdot A''iB'$

Now, A1-model structures validate all members of **A** excepting A1; and, analogously, A2-model structures invalidate but axiom A2.

Note. Vieru-type model structures seem to require that the following stronger constraint

1.2.5. $J_{m'} = J_{m''}$ for any m', m'' in F

is necessary to account for expressions like, e.g., $A'aB''$; indeed, 1.2.5 entails that if A' or B'' are possible term extension at a certain model m', then they shall also be possible term-extensions at any model m''. Then it makes sense to write $A'aB''$.

Let's have a moment's reflection on the meaning of an expression like $A'aB''$. The semantical strategy involved in the work with A3—A8-model structures could be described as follows: it makes sense to compare at any model, say m the extensions of any terms A, B: if $A_m \subseteq B_m$, then the truth-value at m of the expression AaB is 1, and it is 0 if $A_m \subseteq B_m$ does not hold at m a.s.o. However, A1- and A2- model structures ask for a stronger semantical strategy, to the effect that it makes sense to compare the extensions of any terms A, B at different models, e.g., that it is not meaningless to compare the extension A_m of A at m with the extension B_m of B at m' and ask if relation $A_m \subseteq B_m$ holds. Thus, while on the first strategy, $A'aB'$ states that: at m' all A is B, on the later one $A'aB''$ is definable neither at model m', nor at model m''; it might rather be asserted that *at* $F = \{m', m''\}$, all A 's-at-m$'$ are B 's-at-m''.

The meaning of $A'aB''$ is then that the extension of A at m' is contained in the extension of B at m''. Suppose that m' is the (syllogistical description of the) actual world. By 1.3.6, $v'(AaB) = 1$ iff for any m'' in F, $A'' \subseteq B'$. The meaning of this phrase is that the actual extension of B contains the extension of A at any other world (=model). Thus: any A, whatever being an A might mean (at any world) is one of *these* actual B's. In the present context, "B" is accounted for as a rigid designator, while "A" is thought of as a nonrigid one.

Obviously, by virtue of constraints like 1.1.1 or 1.2.5 on members of F (i.e. on components of the members of F), A1-and A2-model structures involve *de re* commitments.